# $\gamma\gamma \to \pi^0\pi^0$

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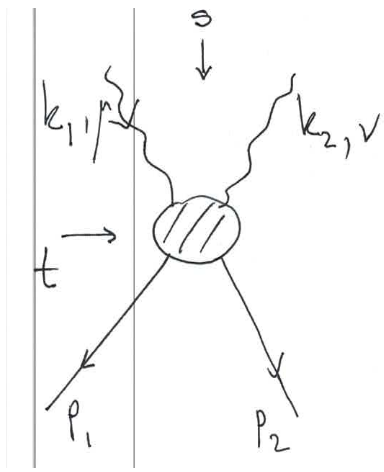
## 1 The amplitude $\gamma^{(*)}\gamma^{(*)} \to \pi^0\pi^0$



The relevant tensor is:

$$V_{\mu\nu} \equiv = \langle p_1, p_2 \mid T(J_\mu(x)J_\nu(y)) \mid 0\rangle \tag{1}$$

where $J_\mu$ is the EM current. Fourier transforming in $x$ and $y$ with momenta $k_1$ and $k_2$ respectively, we can write the most general form for $V_{\mu\nu}$ which respects all symmetries:

$$V_{\mu\nu} = \sum_{i=1}^{5} A_i(s,t,u)T^i_{\mu\nu} \tag{2}$$

where $s,\ t,\ u$ are Mandelstam invariants and the tensor basis which respects gauge invariance is:

$$
\begin{aligned}
T^1_{\mu\nu} &= k_{1\,\nu}\, k_{2\,\mu} - g_{\mu\nu} k_1 \cdot k_2 \\
T^2_{\mu\nu} &= k_{1\,\mu}\, k_{1\,\nu} - g_{\mu\nu} k_1^2 + \frac{1}{k_2 \cdot P}(k_{2\,\mu}\, k_1^2 - k_{1\,\mu}\, k_1 \cdot k_2) \\
T^3_{\mu\nu} &= k_{2\,\mu}\, k_{2\,\nu} - g_{\mu\nu} k_2^2 + \frac{1}{k_1 \cdot P}(k_{1\,\nu}\, k_2^2 - k_{2\,\nu}\, k_1 \cdot k_2) \\
T^4_{\mu\nu} &= P_\mu P_\nu - \frac{1}{k_1 \cdot k_2}(k_{2\,\mu}\, P_\nu k_1 \cdot P + k_{1\,\nu}\, P_\mu k_2 \cdot P - g_{\mu\nu} k_1 \cdot P k_2 \cdot P) \\
T^5_{\mu\nu} &= k_{1\,\mu} k_{2\,\nu} - \frac{1}{k_1 \cdot k_2}(k_1^2 k_{2\,\mu}\, k_{2\,\nu} + k_2^2 k_{1\,\mu}\, k_{1\,\nu} - g_{\mu\nu} k_1^2 k_2^2)
\end{aligned}
\quad (3)
$$

with $P = p_1 - p_2$, we have:

$$
\begin{aligned}
k_1 \cdot k_2 &= \frac{s}{2} - k_1^2 - k_2^2 \\
k_1 \cdot P &= \frac{1}{2}(u - t + p_1^2 - p_2^2) \\
k_2 \cdot P &= -\frac{1}{2}(u - t + p_2^2 - p_1^2)
\end{aligned}
\quad (4)
$$

In the case $p_1^2 = p_2^2$, $k_1 \cdot P = -k_2 \cdot P = \frac{1}{2}(u - t)$.

Bose symmetry requires that:

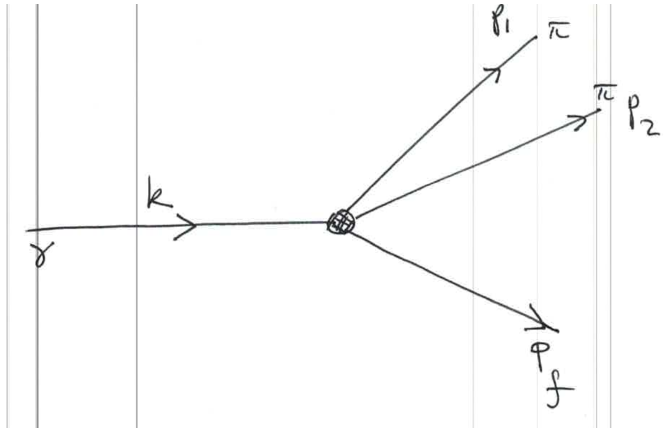
$$
\begin{aligned}
T_{\mu\nu}(P, k_1, k_2) &= T_{\mu\nu}(-P, k_1, k_2) \\
&= T_{\nu\mu}(P, k_2, k_1)
\end{aligned}
\quad (5)
$$

which corresponds also to the exchange $u \leftrightarrow t$. This then implies that:

$$
\begin{aligned}
A_2(s,t,u) &= A_3(s,u,t) \\
A_i(s,t,u) &= A_i(s,u,t) \quad i = 1,4,5
\end{aligned}
\quad (6)
$$

# 2 $\pi^0\pi^0$ photoproduction

## 2.1 Kinematics in Lab frame




Definitions:

$$
\begin{aligned}
\omega &= |\vec{k}| \\
\vec{p}_\pm &= \vec{p}_1 \pm \vec{p}_2, \quad \mathsf{p}_\pm = |\vec{p}_\pm| \\
\vec{p}_f &= \vec{k} - \vec{p}_+, \quad E_f = \sqrt{\vec{p}_f^2 + M^2}
\end{aligned}
\tag{7}
$$

Spherical coordinates: choose $\vec{k}$ in $z$ direction.

$$
\begin{aligned}
\vec{p}_\pm &= \mathsf{p}_\pm(\sin\theta_\pm \cos\phi_\pm, \sin\theta_\pm \sin\phi_\pm, \cos\theta_\pm) \\
E_1^2 &= \frac{1}{4}(\mathsf{p}_+^2 + \mathsf{p}_-^2 + 2\mathsf{p}_+\,\mathsf{p}_-\ \cos\alpha) + \mathsf{M}_\pi^2 \\
E_2^2 &= \frac{1}{4}(\mathsf{p}_+^2 + \mathsf{p}_-^2 - 2\mathsf{p}_+\,\mathsf{p}_-\ \cos\alpha) + \mathsf{M}_\pi^2 \\
\cos\alpha &= \cos\theta_+\ \cos\theta_- + \cos(\phi_+ - \phi_-)\sin\theta_+\ \sin\theta_- \\
\vec{p}_f^{\,2} &= \mathsf{p}_+^2 + \omega^2 - 2\mathsf{p}_+\,\omega\cos\theta_+ \\
E_f^2 &= \vec{p}_f^{\,2} + M^2
\end{aligned}
\tag{8}
$$

so that $E_1 + E_2 = \omega + M - E_f$ depends only on $\mathsf{p}_+$ and $\theta_+$. From the above we get:

$$
E_1 - E_2 = \frac{\mathsf{p}_+\mathsf{p}_-\cos\alpha}{\omega + M - E_f} = \frac{\mathsf{p}_+\mathsf{p}_-\cos\alpha}{E_1 + E_2}
\tag{9}
$$

## 2.2 Differential cross section

$$
\begin{aligned}
d\sigma &= \frac{1}{2(4\pi)^5}\,\frac{|\,\mathcal{M}\,|^2}{\omega M E_1 E_2 E_f}\delta(\omega + M - E_1 - E_2 - E_f)\; d^3p_+ d^3p_- \\
&= \frac{1}{2(4\pi)^5}\frac{|\,\mathcal{M}\,|^2}{\omega M E_1 E_2 E_f}\delta(\omega + M - E_1 - E_2 - E_f)\; \mathsf{p}_+^2\mathsf{p}_-^2\, d\cos\theta_+ d\cos\theta_- d\phi_+ d\phi_- d\mathsf{p}_+ d\mathsf{p}_-
\end{aligned}
$$

using that $\mathsf{p}_+\ \mathsf{p}_-\ \cos\alpha = \vec{p}_+\cdot\vec{p}_- = E_1^2 - E_2^2$, we obtain:

$$
\delta(\omega + M - E_1 - E_2 - E_f) = 4\frac{E_1 E_2 \mathsf{p}_-}{(E_1+E_2)\,|\,\mathsf{p}_-^2 - (E_1 - E_2)^2\,|}\delta(\mathsf{p}_- - \bar{\mathsf{p}}_-) \tag{10}
$$

where

$$
\begin{aligned}
\bar{\mathsf{p}}_- &= \frac{(E_1+E_2)\sqrt{(E_1+E_2)^2 - \mathsf{p}_+{}^2 - 4M_\pi^2)}}{\sqrt{(E_1+E_2)^2 - \mathsf{p}_+^2\cos^2\alpha}} \\
&= \frac{\sqrt{\mathsf{p}_+{}^2 + W_{\pi\pi}}\sqrt{W_{\pi\pi} - 4M_\pi^2}}{\sqrt{W_{\pi\pi} + \mathsf{p}_+{}^2\sin^2\alpha}}
\end{aligned} \tag{11}
$$

Here we defined the squared $\pi\pi$ invariant mass:

$$
\begin{aligned}
W_{\pi\pi} &= (E_1+E_2)^2 - \mathsf{p}_+^2 = 2(\omega^2 + M^2 + \omega M) - 2\omega \mathsf{p}_+\ \cos\theta_+ - 2(\omega+M)E_f \\
&= \left(-\sqrt{M^2 + \mathsf{p}_+^2 - 2\mathsf{p}_+\omega\cos\theta_+ + \omega^2} + M + \omega\right)^2 - \mathsf{p}_+^2
\end{aligned} \tag{12}
$$

The diff cross section then becomes:

$$
\begin{aligned}
d\sigma &= \frac{2}{(4\pi)^5}\ \frac{|\,\mathcal{M}\,|^2}{\omega M E_f (E_1+E_2)\,|\,\bar{\mathsf{p}}_-^2 - (E_1-E_2)^2\,|}\ \mathsf{p}_+^2\bar{\mathsf{p}}_-^3\ d\cos\theta_+ d\cos\theta_- d\phi_+ d\phi_- d\mathsf{p}_+ \\
&= \frac{2}{(4\pi)^5}\ \frac{|\,\mathcal{M}\,|^2}{\omega M E_f\,|\,(E_1+E_2) - \frac{\mathsf{p}_+^2\cos^2\alpha}{E_1+E_2}\,|}\ \mathsf{p}_+^2\bar{\mathsf{p}}_-\ d\cos\theta_+ d\cos\theta_- d\phi_+ d\phi_- d\mathsf{p}_+
\end{aligned} \tag{13}
$$

where we can use:

$$
\begin{aligned}
E_1 + E_2 &= \omega + M - E_f \\
(E_1 - E_2)^2 &= (E_1+E_2)^2 - 4E_1E_2 \\
E_1E_2 &= \sqrt{M_\pi^4 + \frac{1}{2}M_\pi^2(\mathsf{p}_+^2 + \mathsf{p}_-^2) + \frac{1}{4}(\mathsf{p}_+^4 + \mathsf{p}_-^4 - \mathsf{p}_+^2\mathsf{p}_-^2\cos(2\alpha))}
\end{aligned} \tag{14}
$$

It is convenient to express the cross section in terms of the invariant mass squared of the two pion system, where $W_{\pi\pi} > 4M_\pi^2$ and

$$
d\mathsf{p}_+ = \frac{E_f}{2(\mathsf{p}_+(\omega+M) - \omega(E_1+E_2)\cos\theta_+)}dW_{\pi\pi} \tag{15}
$$

One can then write Eq(10) as:

$$\bar{\mathsf{p}}_- = \frac{(E_1+E_2)\sqrt{W_{\pi\pi}-4M_\pi^2}}{\sqrt{W_{\pi\pi}+\mathsf{p}_+^2\sin^2\alpha}} \tag{16}$$

With some work one can replace everywhere $\mathsf{p}_+$ in terms of $W_{\pi\pi}$ using Eq. (12). For this, at a given $\omega$ and $\theta_+$, one needs that:

$$W_{\pi\pi}^2 - 4W_{\pi\pi}(M(M+\omega)+\omega^2\sin^2\theta_+) + 4M^2\omega^2 > 0 \tag{17}$$

and one gets:

$$\mathsf{p}_+ = \frac{\omega\cos\theta_+(2M\omega+W_{\pi\pi}) \pm (M+\omega)\sqrt{-4M^2\left(W_{\pi\pi}-\omega^2\right)-4MW_{\pi\pi}\omega+2W_{\pi\pi}\omega^2\cos 2\theta_+ + W_{\pi\pi}\left(W_{\pi\pi}-2\omega^2\right)}}{2(M+\omega)^2-2\omega^2\cos^2\theta_+} \tag{18}$$

The next step is to determine the physical domain of integration in the angles and $W_{\pi\pi}$. This is being worked out still.

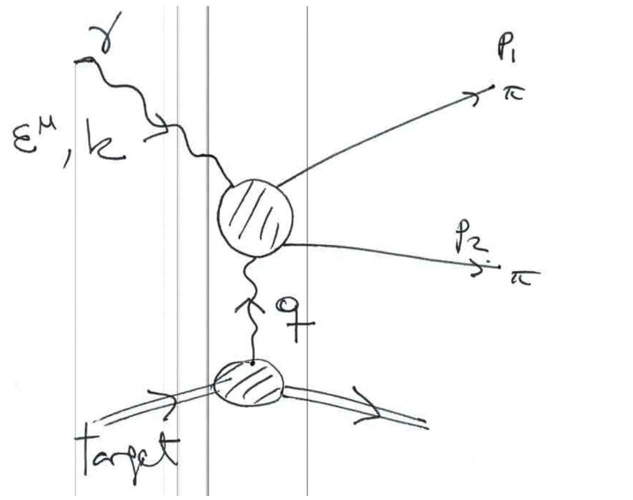
Also, one should find which angular variables are the most convenient to use. This requires that we know in detail the scattering amplitude's angular dependencies in order to make the choice.

## 2.3 Forward limit

We will be interested in the limit of large $\mathsf{p}_+$, small $\theta_+$ and small to moderate $W_{\pi\pi}$, which implies also small $\alpha$.. This also implies that We also want the limit of large $\omega$. In that limit we have:

$$\begin{aligned}
\bar{\mathsf{p}}_- &= \frac{(E_1+E_2)\sqrt{W_{\pi\pi}-4M_\pi^2}}{\sqrt{W_{\pi\pi}+\bar{\mathsf{p}}_+^2\sin^2\alpha}} \\
\bar{\mathsf{p}}_+ &= \frac{\omega(W_{\pi\pi}+2M\omega)+(M+\omega)\sqrt{W_{\pi\pi}(W_{\pi\pi}-4M\omega)+4M^2(\omega^2-W_{\pi\pi})}}{2M(M+2\omega)} \\
&= \omega - \frac{W_{\pi\pi}}{2\omega} - \frac{W_{\pi\pi}^2}{8M\omega^2} - \frac{W_{\pi\pi}^2\left(2M^2+W_{\pi\pi}\right)}{16M^2\omega^3} + \cdots \\
d\bar{\mathsf{p}}_+ &= \left(\frac{1}{2\omega} + \frac{W_{\pi\pi}}{4M\omega^2} + \frac{W_{\pi\pi}\left(\frac{3W_{\pi\pi}}{M^2}+4\right)}{16\omega^3}\right) dW_{\pi\pi} \\
E_f &= \frac{W_{\pi\pi}^3}{16M^2\omega^3} + \frac{W_{\pi\pi}^2}{8M\omega^2} + M \\
E_1+E_2 &= \omega - \frac{W_{\pi\pi}^2}{8M\omega^2} - \frac{W_{\pi\pi}^3}{16M^2\omega^3}
\end{aligned} \tag{19}$$

# 3 Primakoff amplitude and cross section




The scattering amplitude is given by the general expression:

$$\mathcal{M} = \epsilon^\mu T_{\mu\nu}(k, q, p_-) \frac{1}{Q^2} J^\nu \tag{20}$$

$T_{\mu\nu}$ is the Compton tensor, $Q^2 = -q^2$, and the target's EM current in the Lab frame we will neglect the spin of the target, and therefore we only care about the its charge:

$$J^\mu = g^{\mu 0} Z e F(Q^2); \text{ note that we still need to use } q_\nu J^\nu = 0 \tag{21}$$

where $F(Q^2)$ is the charge FF of the target.

Since we are interested in the region of the Primakoff peak, first we approximate the amplitude by using the Compton tensor in the limit of real Compton scattering. This is then directly obtained from the result provided by Bellucci et al. which will be valid for the small $W_{\pi\pi}$ regime. Later I will work out a more detailed analysis where the virtuality $Q^2$ is also included in the Compton tensor, and we will also need to give the amplitude for intermediate values of $W_{\pi\pi}$ (works of Oller and of Pennington).

For a scalar particle the (virtual) Compton tensor is written in terms of five transverse tensors (see Bakker and Ji, Few-Body Syst DOI 10.1007/s00601-016-1172-3)

$$\begin{aligned}
T^{\mu\nu} &= \sum_{n=1}^{5} A_n(s,t,u) T_n^{\mu\nu} \\
T_1^{\mu\nu} &= k \cdot q \; g^{\mu\nu} - k^\nu q^\mu \\
T_2^{\mu\nu} &= (k^\mu k^\rho - k^2 \; g^{\mu\rho})(q_\rho q^\nu - q^2 \; g^\nu_\rho) \\
T_3^{\mu\nu} &= (p_-^\mu k^\rho - p_- \cdot k \; g^{\mu\rho})(p_-^\nu q_\rho - p_- \cdot q \; g^\nu_\rho) \\
T_4^{\mu\nu} &= (p_-^\mu k^\rho - p_- \cdot k \; g^{\mu\rho})(q_\rho q^\nu - q^2 \; g^\nu_\rho) + (k^\mu k^\rho - k^2 \; g^{\mu\rho})(p_-^\nu \, q_\rho - p_- \cdot q \; g^\nu_\rho) \\
T_5^{\mu\nu} &= (k^\mu k^\rho - k^2 \; g^{\mu\rho}) p_{-\rho} \, p_-^\sigma (q_\sigma q^\nu - q^2 \; g^\nu_\sigma)
\end{aligned} \tag{22}$$

Here, using $k^2 = 0$:

$$\begin{aligned} s &= p_+^2 = W_{\pi\pi} \\ u - t &= 2k \cdot p_- = 2\omega(E_1 - E_2 - \mathsf{p}_- \cos\theta_-) \\ s + t + u &= 2M_\pi^2 + q^2 \end{aligned} \tag{23}$$

which allow us to write:

$$\begin{aligned} q \cdot p_- &= -k \cdot p_- = \frac{1}{2}(t - u) \\ k \cdot q &= \frac{1}{2}(W_{\pi\pi} - q^2) \end{aligned}$$

(24)

In the limit $k^2 = q^2 = 0$:

$$\begin{aligned} T_1^{\mu\nu} &= k \cdot q \; g^{\mu\nu} - k^\nu q^\mu \\ T_2^{\mu\nu} &= k \cdot q \; k^\mu q^\nu \\ T_3^{\mu\nu} &= (p_-^\mu k^\rho - p_- \cdot k \; g^{\mu\rho})(p_-^\nu q_\rho - p_- \cdot q \; g_\rho^\nu) \\ T_4^{\mu\nu} &= (p_-^\mu k \cdot q - p_- \cdot k \; q^\mu) q^\nu + k^\mu (p_-^\nu \; k \cdot q - p_- \cdot q \; k^\nu) \\ T_5^{\mu\nu} &= k \cdot p_- \; q \cdot p_- \; k^\mu q^\nu \end{aligned} \tag{25}$$

where for real photons only $T_1$ and $T_3$ will contribute to the amplitude after contracting with photon polarizations. The tensor we need in this limit, eliminating terms that are proportional to $k^\mu$ due to transversity with the incoming photon polarization:

$$T_{\mu\nu} = A(W_{\pi\pi}, t, u)(\frac{1}{2} W_{\pi\pi} g_{\mu\nu} - k_\nu q_\mu) \tag{26}$$

$$\begin{aligned} &+ 2B(W_{\pi\pi}, t, u)((W_{\pi\pi} - q^2) p_{-\mu} p_{-\nu} - 2(k \cdot p_- \; q_\mu p_{-\nu} + q \cdot p_- k_\nu p_{-\mu} - g_{\mu\nu} k \cdot p_- \;\; q \cdot p_-)) \\ &+ C(W_{\pi\pi}, t, u)(W_{\pi\pi} \, p_-^\mu - 2 \, p_- \cdot k \; q^\mu) q^\nu \end{aligned} \tag{27}$$

where $p_- = p_1 - p_2$. For the case where one contracts the Compton tensor with a conserved current as it is the case hare, namely $T^{\mu\nu} J_\nu$ where $J_\nu$ is the target electric current. Since $q^\mu J_\mu = 0$, the term $C(W_{\pi\pi}, t, u)$ does not contribute.

The low energy theorem for Compton scattering gives the following constraints:

$$\begin{aligned} \frac{\alpha}{2M_\pi}(A + 16 M_\pi^2 B)|_{W_{\pi\pi}=0, t=M_\pi^2} &= \alpha_\pi \\ -\frac{\alpha}{2M_\pi} A|_{W_{\pi\pi}=0, t=M_\pi^2} &= \beta_\pi \end{aligned} \tag{28}$$

where $\alpha_\pi$ $\beta_\pi$ are the electric and magnetic polarizabilities respectively.

For the functions $A$ and $B$ there are low energy results in ChPT (Bellucci et al) at two loops. The results are as follows:

$$
\begin{aligned}
A(s,t,u) &= 4\frac{G_\pi(s)}{sF_\pi^2}(s - M_\pi^2) + U_A + P_A \\
B(s,t,u) &= U_B + P_B
\end{aligned} \tag{29}
$$

where the functions and polynomials $U$ and $P$ are given in Bellucci's et al., see Appendix A:

$$
G_\pi(s) = -\frac{1}{(4\pi)^2}\left(1 + 2\frac{M_\pi^2}{s}\int_0^1 \frac{dx}{x}\log(1 - \frac{s}{M_\pi^2}x(1-x))\right) \tag{30}
$$

Use the integral in terms of dilogarithm functions:

$$
\int_0^1 \frac{dx}{x}\log(1 - Ux(1-x)) = -\mathrm{Li}_2\left(\frac{1}{2}\left(U - \sqrt{U-4}\sqrt{U}\right)\right) - \mathrm{Li}_2\left(\frac{1}{2}\left(U + \sqrt{U-4}\sqrt{U}\right)\right) \tag{31}
$$

where in our case $U$ must be taken to have an imaginary part $+i\epsilon$. At low energy $W_{\pi\pi} < (0.4\mathrm{GeV})^2$ the $t$ dependence of the amplitudes $A$ and $B$ is very small and can be neglected. We however should later consider also the effects of $Q^2 > 0$ and check that claim.

## 3.1 Amplitude squared in Lab frame

$$
\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{Q^4}Z^2e^2F^2(Q^2)\,|A(s,t,u)\,\omega\,\epsilon\cdot q \\
&- 2B(s,t,u)(((W_{\pi\pi} + Q^2)(E_1 - E_2) + 2\,\omega\,k\cdot p_-)\epsilon\cdot p_- - 2(E_1 - E_2)k\cdot p_-\ \epsilon\cdot q) \\
&= \frac{1}{Q^4}Z^2e^2F^2(Q^2)\,|K_1\ \epsilon\cdot q + K_2\ \epsilon\cdot p_-|^2
\end{aligned} \tag{32}
$$

where

$$
\begin{aligned}
K_1 &= \omega A(s,t,u) + 4B(s,t,u)(E_1 - E_2)k\cdot p_- \\
K_2 &= -2B(s,t,u)((W_{\pi\pi} + Q^2)(E_1 - E_2) + 2\omega k\cdot p_-)
\end{aligned} \tag{33}
$$

The relevant products we need to use are:

$$
\begin{aligned}
\epsilon\cdot q &= -\vec{\epsilon}\cdot\vec{p}_+ = -\mathsf{p}_+\sin\theta_+\cos\phi_+ \\
\epsilon\cdot p_- &= -\mathsf{p}_-\sin\theta_-\cos\phi_- \\
q^2 &= -Q^2 = W_{\pi\pi} - 2\omega(E_1 + E_2 - \mathsf{p}_+\ \cos\theta_+) \\
k\cdot p_- &= \omega(E_1 - E_2 - \mathsf{p}_-\cos\theta_-) \\
q\cdot p_- &= -k\cdot p_- = -\omega(E_1 - E_2 - \mathsf{p}_-\cos\theta_-) \\
p_-^2 &= 4M_\pi^2 - W_{\pi\pi}
\end{aligned} \tag{34}
$$

In the case of unpolarized photon beam we get:

$$|\mathcal{M}|^2 = \frac{1}{2}\frac{1}{Q^4}Z^2e^2F^2(Q^2)\left(-|K_1|^2p_-^2 + |K_2|^2Q^2 + 2Re(K_1K_2^*)\, p_-\cdot k\right) \tag{36}$$

$$\begin{aligned}
|\mathcal{M}|^2 &= \frac{1}{Q^4}Z^2e^2F^2(Q^2)\,(A\,\omega\,q^\mu - 2B\,(E_1-E_2)((s+Q^2+q\cdot p_-)p_-^\mu - 2k\cdot p_-\,q^\mu)\\
&\times\ (A^*\,\omega\,q_\mu - 2B^*\,(E_1-E_2)((s+Q^2+q\cdot p_-)p_{-\mu} - 2k\cdot p_-\,q_\mu)\\
&= \frac{e^2Z^2F\left(Q^2\right)^2}{Q^4}\Bigg(Q^2\omega^2\Big(-\mid A\mid^2 - 16\mid B\mid^2 (E_1-E_2)^2\\
&\times\ \Big((E_1+E_2)\cos(\theta_-)\sqrt{\frac{s-2{M_\pi}^2}{{\mathsf{p}_+}^2\sin^2(\alpha)+s}} - E_1 + E_2\Big)^2\Big)\\
&+\ \left(\omega(E_1+E_2)\cos(\theta_-)\sqrt{\frac{s-2{M_\pi}^2}{{\mathsf{p}_+}^2\sin^2(\alpha)+s}} - E_1\omega + E_2\omega + Q^2 + s\right)\\
&\times\ \left(4Re(AB^*)\omega^2(E_1-E_2)\left(-(E_1+E_2)\cos(\theta_-)\sqrt{\frac{s-2{M_\pi}^2}{{\mathsf{p}_+}^2\sin^2(\alpha)+s}} + E_1 - E_2\right)\right.\\
&+\ 16\mid B\mid^2\omega^2(E_1-E_2)^2\left((E_1+E_2)\cos(\theta_-)\sqrt{\frac{s-2{M_\pi}^2}{{\mathsf{p}_+}^2\sin^2(\alpha)+s}} - E_1 + E_2\right)^2\\
&+\ 4\mid B\mid^2(E_1-E_2)^2\left(\frac{(E_1+E_2)^2\left(2{M_\pi}^2 - s\right)}{{\mathsf{p}_+}^2\sin^2(\alpha)+s} + (E_1-E_2)^2\right)\\
&\times\ \left.\left(\omega(E_1+E_2)\cos(\theta_-)\sqrt{\frac{s-2{M_\pi}^2}{{\mathsf{p}_+}^2\sin^2(\alpha)+s}} - E_1\omega + E_2\omega + Q^2 + s\right)\right)\Bigg)
\end{aligned} \tag{37}$$

## 3.2 Amplitudes $A$ and $B$ for simulation

We need to have a parametrization which for now gives a sufficiently realistic description for carrying out simulations. We present here a model for $A(s,t,u)$ and $B(s,t,u)$ based on dispersion theory to take into account the FSI of the pions with addition of t- and u-channel exchanges of resonances. At present there are many works with increasing level of rigor, with the latest ones being quite complicated. So, for a first approximation we adopt here one of the early models by Donoghue and Holstein (1994).

The Donoghue-Holstein model: premises of the model: 1) only S-wave $\pi\pi$ FSI are considered, which is accurate at small $W_{\pi\pi}$. 2) vector (also axial vector for the case of $\pi^+\pi^-$)

exchanges in t- and u- channels of the $\gamma\gamma \to \pi\pi$ reaction. The latter give the t- and u-dependency. For the neutral pions the model gives:

$$\begin{aligned} s\, A(s,t,u) &= -\frac{2}{3}(f_0(s) - f_2(s)) + \frac{2}{3}(p_0(s) - p_2(s)) - \frac{s}{2}\sum_{V=\rho,\omega} R_V(\frac{t+M_\pi^2}{t-M_V^2} + \frac{u+M_\pi^2}{u-M_V^2}) \\ B(s,t,u) &= -\frac{1}{8}\sum_{V=\rho,\omega} R_V(\frac{1}{t-M_V^2} + \frac{1}{u-M_V^2}) \end{aligned} \tag{38}$$

Here the subindices 0 and 2 indicate the Isospin state of the pion pair. $f_I(s)$ admit a dispersive representation for which we need the $I = 0$ and 2 S-wave $\pi\pi$ phase shifts. In the case of the neutral pions we only have $\rho_0$ and $\omega$ exchanges, where here $R_V$ indicates the coupling in the vertex $V \to \gamma\pi^0$. Here we use:

$$R_V = \frac{6M_V^2}{\alpha}\frac{\Gamma(V \to \pi\gamma)}{(M_V^2 - M_\pi^2)^3} \tag{39}$$

The functions $p_I(s)$ are known (given below by the model of resonance t- and u-channel exchanges), are real for positive $s$ and reproduce the smae discontinuity as $f_I(s)$ for negative $s$ .

The hard problem is to implement the dispersive representation for $f_I(s)$. We will neglect inelasticities in the $\pi\pi$ FSI. The DH model implements the approach of Morgan and Pennington who write a twice subtracted dispersion relation for the combination $(f_I(s) - p_I(s))/\Omega_I(s)$, which has only a discontinuity for $s > 4M_\pi^2$ and is otherwise analytic everywhere. $\Omega_I(s)$ is the Omnès function given in terms of the corresponding $\pi\pi$ phase shift:

$$\begin{aligned} \Omega_I(s) &= \exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\phi_I(s')}{s'-s}\frac{ds'}{s'}\right) \\ \Omega_I(s > 4M_\pi^2) &= e^{i\phi_I(s)}\exp\left(\frac{s}{\pi}\int_{4M_\pi^2}^{\infty}\frac{\phi_I(s') - \phi_I(s)}{s'-s}\frac{ds'}{s'} + \frac{\phi_I(s)}{\pi}\log\frac{4M_\pi^2}{s-4M_\pi^2}\right) \end{aligned} \tag{40}$$

where the phase $\pi$ is related to the corresponding $\pi\pi$ phase shift: $\phi_0(s) = \theta(M - \sqrt{s})\delta_0^0(s) + \theta(\sqrt{s} - M)(\pi - \delta_0^0(s))$, where $M$ is the mass of the $f_0$ resonance. For $I = 2$ one can take $\phi_2(s) = \delta_0^2(s)$.
We have that:

$$f_I(s) = p_I(s) + \Omega_I(s)\left(c_I + d_I\, s - \frac{s^2}{\pi}\int_{4M_\pi^2}^{\infty} p_I(s')\Im(\Omega_I^{-1}(s'))\frac{ds'}{(s'-s)s'^2}\right) \tag{41}$$

Finally, the functions $p_I(s)$ are as follows:

$$
\begin{aligned}
p_I(s) &= f_I^{\mathrm{Born}}(s) + p_I^A(s) + p_I^\rho(s) + p_I^\omega(s) \\
p_0^A(s) = p_2^A(s) &= \frac{L_9^r + L_{10}^r}{F_\pi^2}\left(s + \frac{M_A^2 - M_\pi^2}{\beta(s)} \log \frac{1+\beta(s)+s_A/s}{1-\beta(s)+s_A/s}\right) \\
p_0^\rho(s) &= \frac{3}{2} R_\rho \left(\frac{M_\rho^2}{\beta(s)} \log \frac{1+\beta(s)+s_\rho/s}{1-\beta(s)+s_\rho/s}\right) \\
p_2^\rho(s) &= 0 \\
p_0^\omega(s) = -\frac{1}{2} p_0^\omega(s) &= -\frac{1}{2} R_\omega \left(\frac{M_\omega^2}{\beta(s)} \log \frac{1+\beta(s)+s_\omega/s}{1-\beta(s)+s_\omega/s} - s\right) \qquad (42)
\end{aligned}
$$

where $L_9^r$ and $L_{10}^r$ are the renormalized $\mathcal{O}(p^4)$ LECs and given by: $L_9^r + L_{10}^r = 1.43 \pm 0.27 \times 10^{-3}$, and:

$$
\begin{aligned}
s_i &= 2(M_i^2 - M_\pi^2) \\
R_\omega &= 1.35/GeV^2; \quad R_\rho = 0.12/GeV^2 \\
\beta(s) &= \sqrt{\frac{s - 4M_\pi^2}{s}} \qquad (43)
\end{aligned}
$$

Numerical implementation: we need a simple functional parametrization of the phase shifts (in progress).

## 3.3 Parametrization of the S-wave $\pi\pi$ phase shifts

The following figures show the present knowledge of the relevant phase shifts:

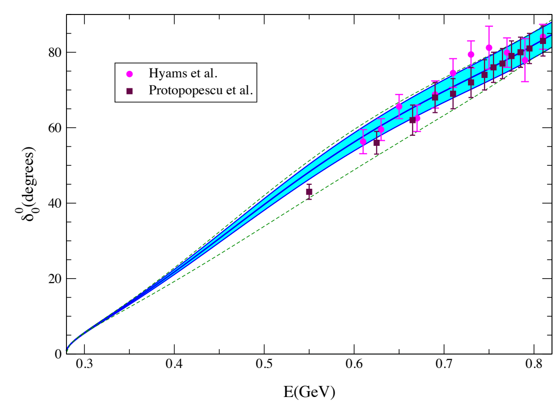
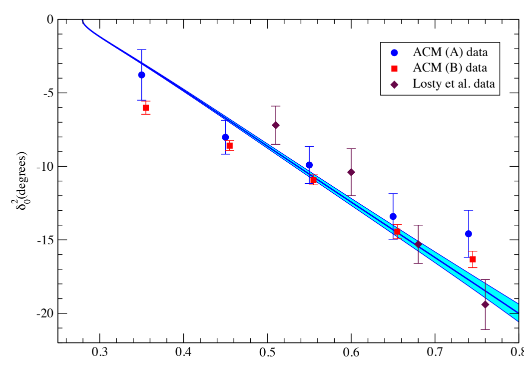



Figure 7: $I = 0$ $S$–wave phase shift. The full line results with the central values of the scattering lengths and of the experimental input used in the Roy equations. The shaded region corresponds to the uncertainties of the result. The dotted lines indicate the boundaries of the region allowed if the constraints imposed by chiral symmetry are ignored [6]. The data points are from refs. [49] and [50].



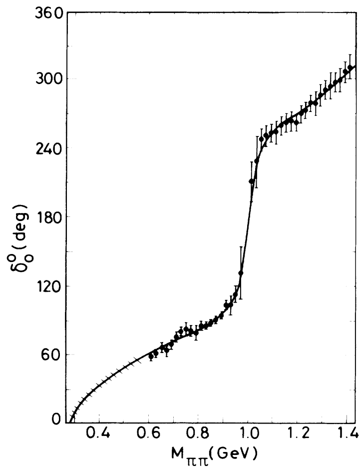



FIG. 3. The $I=0$ $S$-wave phase shift $\delta_0^0$ for $\pi\pi$ scattering (denoted $\delta_{11}$ in the text) from the CERN-Munich group (Ref. 29). The hatched band represents the continuation down to threshold provided by the Roy equations (Ref. 33). The curve shows a fit typical of all our solutions.

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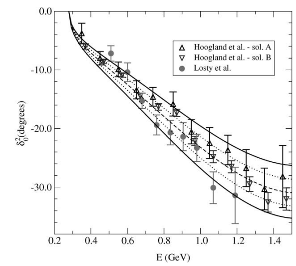



Fig. 5. Different data sets for the S-wave in the $I=2$ channel and curves that we have used as input in the Roy equation analysis.

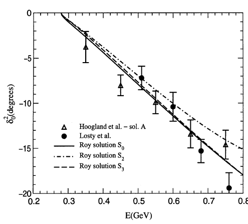



Fig. 11. Comparison of our Roy solutions with the data on $\delta_0^2$ obtained by the ACM collaboration [54] and by Losty et al. [52]. The full, dash–dotted and dashed lines correspond to the points $S_0$, $S_2$ and $S_3$ in Fig. 7.

We fit to the phase shift data the following forms:

$$\delta_0^I(s) = \arcsin\left(\frac{\Gamma_I}{2\sqrt{(\sqrt{s}-M_I)^2+\frac{\Gamma_I^2}{4}}}\right) + \sum_{n=0}^{N} a_n(\sqrt{s})^n \tag{44}$$

where we include one single resonance for each $I=0,2$.

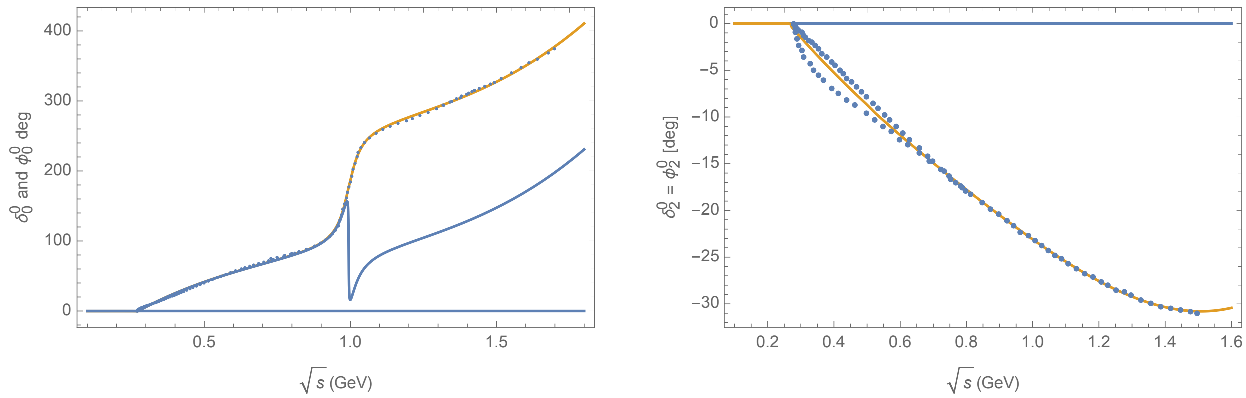
For the available data we need only up to $N=3$ for $I=0$, with the result:

$$M_0 = 0.994GeV; \quad \Gamma_0 = 0.0624GeV$$
$$a_0 = -1.439; \quad a_1 = 6.461/GeV; \quad a_2 = -5.529/GeV^2; \quad a_3 = 2.022/GeV^3 \tag{45}$$

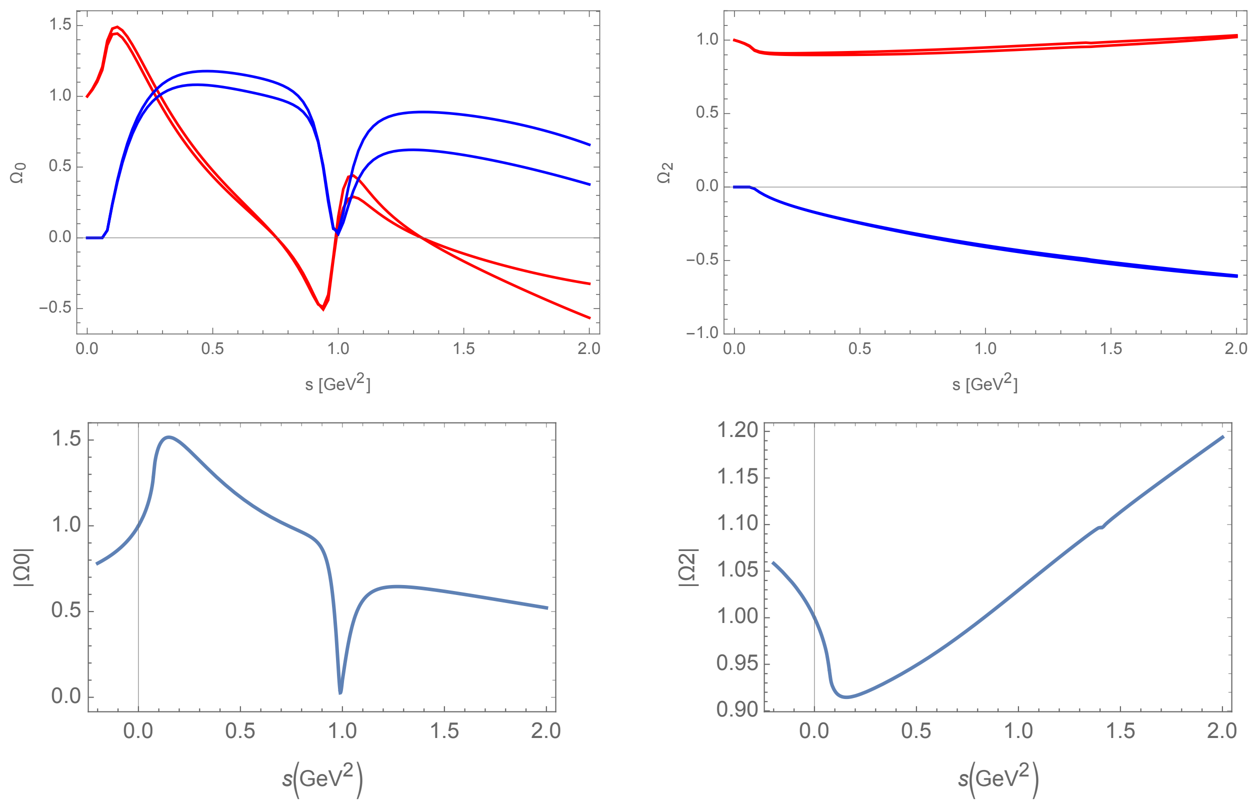
For the case $I=2$ one finds that the resonance term is not needed at all and a good fit is provided with $N=3$ with the result:

$$a_0 = -0.878; a_1 = -0.611/GeV; \quad a_2 = -0.083/GeV^2; \quad a_3 = 0.115/GeV^3 \tag{46}$$

The figure shows the parametrized phase shifts along with the corresponding phase $\phi_I^0$.

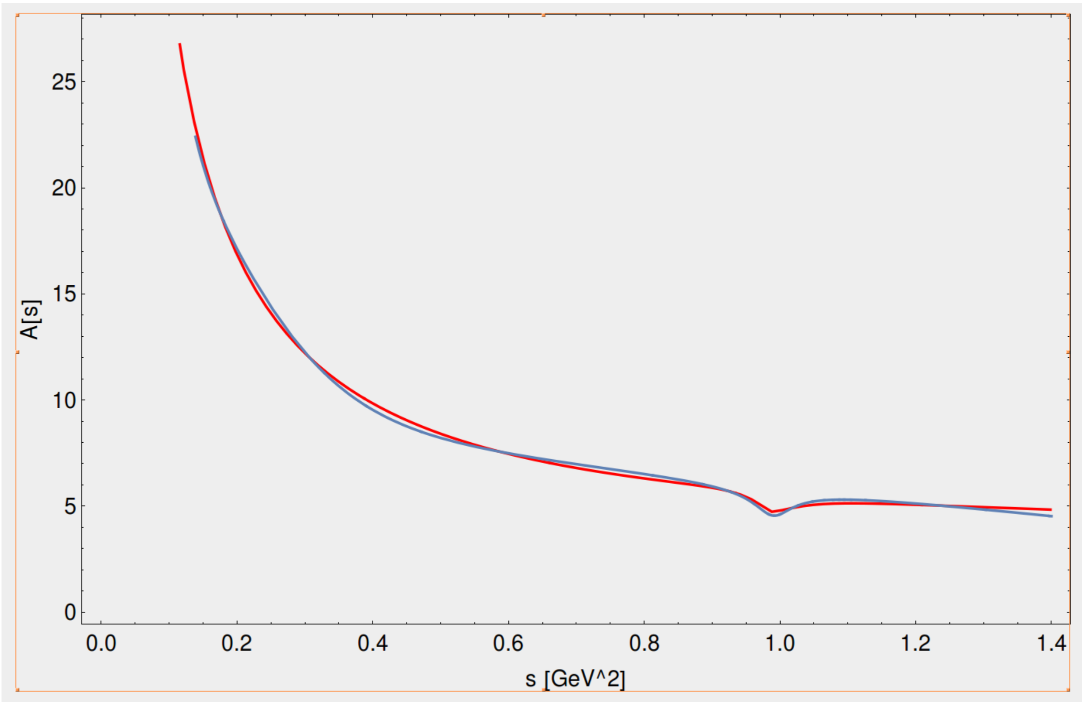



The resulting Omnès functions are shown in the figure, where the red curves show the real part and blue the imaginary ones.:




We used the integration in $s'$ up to 1.4 and 2.4 $GeV^2$ to see how sensitive is the Omneès function to the high energy inputs of the phase shifts. In red the real parts and in blue the imaginary parts.

The amplitude $A(s,t,u)$ in the limit of real photons, and in the approach followed here of keeping only the S-wave, becomes only a function of $s$. Using the results obtained here and and a fit to the available cross section (see the next section), one obtains that $A$ is real and given by the figure below. We are working on a parametrization at this point.




Blue: parametrization

Still working on improving parametrization of the Amplitude A

Result of one parametrization: since we are using the S-wave approximation, there is no dependency on $t$, so:

$$
\begin{aligned}
Re\, A_{par}(s) &= -\frac{0.0401449}{|s-(0.99\ -i\ 0.027)|} - 3.42726\ s \\
&+ 0.00399002\ e^{5.90315(s-1.48297)^2} - 0.00202174\ e^{8.10357(s-1.37113)^2} + 0.000320964 e^{9.36221(s-1.32585)^2} \\
&+ 0.000249196\ e^{9.41227(s-1.31682)^2} + 0.000158993\ e^{8.04004(s-1.25463)^2} \\
&+ 531.611\ \tanh(21.5329(s-0.00549155)) - 522.191 \qquad (47)
\end{aligned}
$$

The imaginary part of $A$ is very small so we neglect it for now.

# 4 $\gamma\gamma \to \pi^0\pi^0$ Cross section

For real photons the $\gamma\gamma \to \pi^0\pi^0$ cross section becomes:

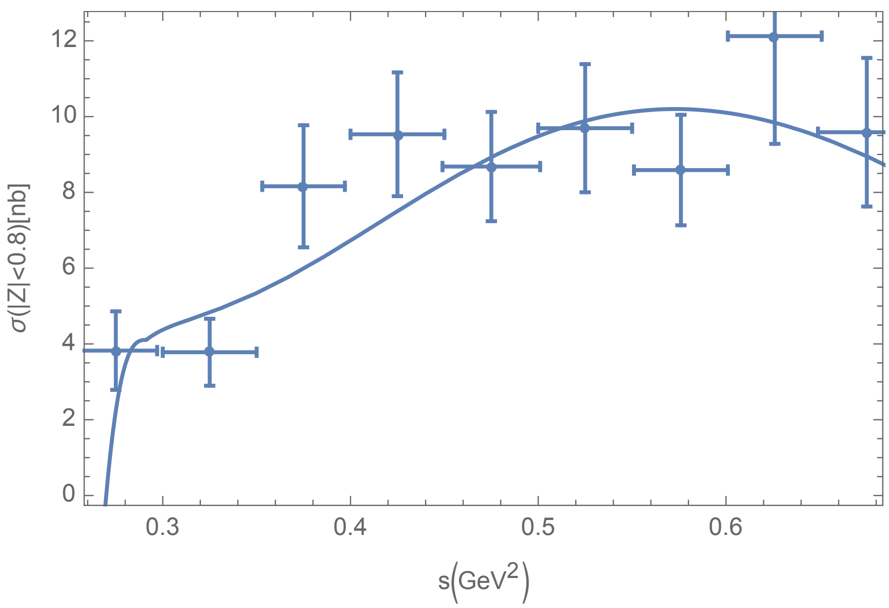
$$
\begin{aligned}
\sigma_{\gamma\gamma\to\pi^0\pi^0}(|\cos\theta| < Z)(s) &= \frac{\pi\alpha_{EM}^2}{s^2}\int_{-Z}^{Z}\frac{1}{4}\sqrt{s(s-4M_\pi^2)} \qquad (48) \\
&\times \ (|\ A(s,t,u)s - M_\pi^2 B(s,t,u)\ |^2 + \frac{1}{s^2}(M_\pi^2 - t\,u)\ |\ B(s,t,u)\ |^2) dz
\end{aligned}
$$

where $z = \cos\theta$ and CM we have: $s + t + u = 2M_\pi^2$, and $t = M_\pi^2 - \frac{s}{2} + \frac{1}{4}\sqrt{s(s-4M_\pi^2)}z$.

We fit to the Cristal Ball data the parameters $c_0,\ d_0,\ c_2,\ d_2$ which give in corresponding units:

$$
\begin{aligned}
c_0 &= -0.585 \\
d_0 &= 3.408 \\
c_2 &= -1.104 \\
d_2 &= 4.222
\end{aligned}
\tag{49}
$$

The result is shown in the figure:




$\gamma\gamma \to \pi^0\pi^0$ cross section integrated for the scattering angle $|\cos\theta| < 0.8$. Data from the Cristal Ball

In progress: a fit to a wider range in $s$ to include if possible the region of the $f_0$

# 5 Possible hadronic exchange background

The possible hadronic t-exchange that can contribute to the $\pi^0\pi^0$ coherent photoproduction will involve $\rho^0$ and $\omega$ exchanges. We need to model this.

# 6 Appendix A

## 6.1 $U_A$ and $P_A$ in ChPT (Bellucci et al)

$$
\begin{aligned}
U_A &= \frac{2}{sF_\pi^4} G_\pi(s) \big((s^2 - M_\pi^2) J_\pi(s) + C(s)\big) + \frac{\ell_\Delta}{24\pi^2 F_\pi^4}(s - M_\pi^2) J_\pi(s) \\
&+ \frac{\ell_2 - 5/6}{144\pi^2 s F_\pi^4}(s - 4M_\pi^2)(H(s) + 4(sG_\pi(s) + 2M_\pi^2(\tilde{G}_\pi(s) - 3\tilde{J}_\pi(s)))d_{00}^2) \\
P_A &= \frac{1}{(4\pi)^2 F_\pi^4}(a_1 M_\pi^2 + a_2 s)
\end{aligned}
\tag{50}
$$

where the constants $a_1$ and $a_2$ need to be fitted, and:

$$
\begin{aligned}
J_\pi(s) &= -\frac{1}{(4\pi)^2}\int_0^1 dx \log(1 - \frac{s}{M_\pi^2}x(1-x)) = \frac{2}{(4\pi)^2}\left(1 - \frac{\sqrt{4 - \frac{s}{{M_\pi}^2}}\tan^{-1}\left(\frac{\sqrt{\frac{s}{{M_\pi}^2}}}{\sqrt{4-\frac{s}{{M_\pi}^2}}}\right)}{\sqrt{\frac{s}{{M_\pi}^2}}}\right) \\
\tilde{J}_\pi(s) &= J_\pi(s) - sJ'_\pi(0) \\
\tilde{G}_\pi(s) &= G_\pi(s) - s\,G'_\pi(0) \\
H_\pi(s) &= (s - 10\,M_\pi^2)\;J_\pi(s) + 6\,M_\pi^2\;G_\pi(s)
\end{aligned}
\tag{51}
$$

and:

$$
\begin{aligned}
C(s) &= \frac{1}{48\pi^2}\bigg(2(\ell_1 - \frac{4}{3})(s - 2M_\pi^2)^2 + \frac{1}{3}(\ell_2 - \frac{5}{6})(4s^2 - 8sM_\pi^2 + 16M_\pi^4) \\
&- 3M_\pi^4\ell_3 + 12M_\pi^2(s - M_\pi^2)\ell_4 - 12sM_\pi^2 + 15M_\pi^4\bigg) \\
d_{00}^2 &= \frac{1}{2}(3\cos^2\theta_{CM} - 1)
\end{aligned}
\tag{52}
$$

where $\theta_{CM}$ is the $\gamma\gamma^* \to \pi\pi$ scattering angle in CM, and the low energy constants $\ell_i$ are known.

Note that the amplitude depends only on $s$ except for the term $d_{00}^2$. It is possible that this term will be entirely irrelevant at low $W_{\pi\pi}$ (need to check).

## 6.2 $U_B$ and $P_B$

$$
\begin{aligned}
U_B &= \frac{\ell_2 - \frac{5}{6}}{288\pi^2 F_\pi^4 s} H_\pi(s) \\
P_B &= \frac{b}{(4\pi F_\pi)^4}
\end{aligned}
\tag{53}
$$

where $b$ is fitted.

# 7 Appendix B

CM kinematics

Useful invariants in Lab frame:

$$\begin{aligned}
q &= \mathsf{p}_+ - k \\
\epsilon^\mu q_\mu &= -\vec{\epsilon}\cdot\vec{q} = -\vec{\epsilon}\cdot\vec{\mathsf{p}}_+ = -\mathsf{p}_+ \sin\theta_+ \cos\phi_+ \\
\epsilon^\mu p_{-\mu} &= -\vec{\epsilon}\cdot\vec{\mathsf{p}}_- = -\mathsf{p}_- \sin\theta_- \cos\phi_- \\
k^\mu J_\mu &= \omega ZeF(Q^2) \\
k^\mu p_{+\mu} &= k^\mu q_\mu = \omega(E_1 + E_2 - \mathsf{p}_+ \cos\theta_+) \\
q^\mu p_{+\mu} &= s - k^\mu p_{+\mu} \\
Q^2 &= -s + 2\,k^\mu p_{+\mu} = -s + 2\omega((E_1+E_2) - \mathsf{p}_+\cos\theta_+) \\
&= 2\omega\mathsf{p}_+(1-\cos\theta_+) + s(\frac{\omega}{\mathsf{p}_+} - 1) - s^2\frac{\omega}{4\mathsf{p}_+^3} + \cdots \\
q^\mu p_{-\mu} &= -k^\mu p_{-\mu} = -\omega(E_1 - E_2 - \mathsf{p}_-\cos\theta_-)
\end{aligned} \tag{54}$$

$$\begin{aligned}
s &= 4\,\omega_{CM}^2 = p_+ \cdot p_+ \\
-\frac{1}{2}\sqrt{s(s-4M_\pi^2)}\cos\theta_{CM} &= k\cdot p_- = \omega(E_1 - E_2 - \mathsf{p}_-\cos\theta_-)
\end{aligned} \tag{55}$$

where we can use:

$$\begin{aligned}
E_1 + E_2 &= \sqrt{s + \mathsf{p}_+^2} \\
E_1 - E_2 &= \frac{\mathsf{p}_+\sqrt{s-4M_\pi^2}\cos\alpha}{\sqrt{s+\mathsf{p}_+^2\sin^2\alpha}}
\end{aligned} \tag{56}$$