Monte Carlo Generation of Electron Angular Distribution for Uncollimated Incoherent Bremsstrahlung

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This note is being distributed as a call for comment and criticism – I am not certain that this procedure is valid, and would like to run it past more experienced Monte Carlo developers.

A. Intended procedure

In thinking about the possibility of generating an angular distribution of electrons corresponding to **collimated** bremsstrahlung, I have concluded that the procedure I originally had in mind contains some self-contradictions. The photon collimation aperture is so small that its boundaries cannot be defined by a sharp angular cutoff, but only as an average over the angular distribution of the incident electron beam, etc. Thus a realistic Monte Carlo calculation for the outgoing bremsstrahlung electrons associated with collimated photons must include not merely an effective electron angular distribution but an event-by-event integral over the photon collimation aperture using the theoretical fully-differential bremsstrahlung cross section, something that I am not prepared to consider doing.

Thus I am limiting myself for the present to what I hope is a realistic calculation for **uncollimated incoherent** bremsstrahlung, for which the electron angular distribution can be calculated and used in the following procedure:

1. Generate the photon energy k and electron energy $E_e = E_0 - k$
2. Generate the electron angle at the radiator using the uncollimated electron angular distribution for the given value of E_e
3. Modify the electron angle and position at the radiator using
   - the incident electron beam ellipse parameters
   - multiple scattering in the radiator, using the energy E_0 for the first ≈half of the radiator and E_e for the second ≈half (using two steps like this appears to be a good approximation to a full integral)
4. Propagate the electron through the beam pipes, quadrupole and dipole to the exit window
5. Allow multiple scattering at the exit window, and propagate the electron to the counter planes, ...

Comment on step 3:
Multiple scattering in the radiator is more significant than the angular distribution of the incident beam, but mostly small compared to the intrinsic bremsstrahlung electron angular distribution which is of order $\theta_{ce} = (m/E_0)(k/E_e)$. This is shown in Table I (at the end of the document.)

B. Angular distribution of electrons from uncollimated bremsstrahlung

To calculate the electron angular distribution I have used formulas for the fully differential cross section $1/Z^2$ $d\sigma/d(k/m)\, d\Omega_e\, d\Omega_\gamma$ (units fm^2/sr^2) for incoherent electron bremsstrahlung in "spectrometer coordinates" as given by Leonard Maximon in his unpublished 1988 Moscow conference paper [Max88]. Back in the period 1989-1998 I wrote programs to integrate this cross section over either electron or photon angles, and verified that they give sensible results for the partially-integrated cross sections $d\sigma/dk\, d\Omega_e$ and $d\sigma/dk\, d\Omega_\gamma$ as well as the fully-integrated cross section $d\sigma/dk$.

For the present calculations I used

- $E_0 = 12$ GeV
- $Z = 6$
- screening using both Thomas-Fermi-Moliere form factors and relativistic Hartree-Fock form factors taken from Hubble and Øverbø [Hub79]. (The results of the two screening calculations do not differ appreciably).
- 8 values of electron energy: $E_e / E_0 = 0.025, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3$ and 0.35
- 34 values of electron angle: $\theta/\theta_{ce} = 0, 0.1, 0.2, ..., 1.0, 1.2, ..., 2.0, 2.5, ..., 5, 6, ..., 10$

For each electron energy and angle, I integrated over the photon angle from 0 to 30 θ_c to give the differential cross section $1/Z^2$ $d\sigma/d(k/m)\, d\Omega_e$ in units of fm^2/sr. (An alternate calculation, integrating "to convergence", gives essentially identical results much more quickly.)

The important qualitative results of the calculation are:

1. The maximum value of the cross section, $1/Z^2$ $d\sigma/d(k/m)\, d\Omega_e(\theta_e=0)$ in fm^2/sr , increases with E_e, scaling approximately as $1/\theta_{ce}^2$

2. When scaled to the $\theta_e=0$ value, $d\sigma/dk\, d\Omega_e$ is very nearly a universal function of θ_e/θ_{ce}, with only a small dependence on E_e/E_0 in the range $0.025 < E_e/E_0 < 0.35$ (see Figure 1, which plots the same data using both linear and logarithmic scales.)

3. The central region, $\theta_e/\theta_{ce} < 0.6$, is approximated fairly well by a Gaussian of $\sigma = 0.55$ (in units of θ_{ce})

4. At larger angles, the angular distribution in any region can be matched reasonably well by taking $d\sigma/d\Omega_e$ as an exponential function. For reasons I discuss below, however, I have chosen to approximate the angular distribution by the function $d\sigma/d\Omega_e = A e^{-bz}/z$, where $z = \theta_e/\theta_{ce}$. Reasonable agreement is found using
   $b =$ 0.5 $0.6 < z < 1.6$
         0.8 $1.6 < z < 5$
   For z>5, the contribution to the angular integral is negligible, so no additional function is required. We can let the upper limit of z go to 10 or any other value >5.

Curves corresponding to the functions given in items 3 and 4 are shown in Figure 1, where it is seen that they agree reasonably well with the angular distribution for all values of E_e/E_0 .

C. Generating the angular distribution in Monte Carlo calculations

Since the overall angular distribution is non-Gaussian, the usual tricks employed for multiple scattering cannot be used here. Instead, we require a method of generating events with angle θ distributed according to an angular distribution $dN/d\Omega$, where $d\Omega = \sin\theta \, d\theta \, d\varphi \rightarrow 2\pi \sin\theta \, d\theta$. Thus the angle-generating distribution must be

$$dN/d\theta = 2\pi \, dN/d\Omega \;\sin\theta \approx 2\pi\,\theta\, dN/d\Omega \qquad (1)$$

For an arbitrary angular distribution $d\sigma/d\Omega = f(\theta)$, the distribution function for the angle θ is given by $dN/d\theta = A\, f(\theta) \sin\theta \approx A\, \theta\, f(\theta)$ for small angles. (In the present problem, θ_e is always less than a few milliradians.)

1. Gaussian distribution

If $f(\theta) = C \exp(-\theta^2/2\sigma^2)$, then

$$dN/d\theta = A\,\theta \exp(-\theta^2/2\sigma^2) = d/d\theta\,[-A\sigma^2 \exp(-\theta^2/2\sigma^2)\,]. \qquad (2)$$

Because $dN/d\theta$ is the derivative of an invertible function, we can generate θ using the inverse transform method (see Particle Data Group, Review of Particle Physics 2014 [RPP14], Section 39.2):

If x is to have the differential probability distribution $g(x)$,
then choose Y = uniform random variate on the appropriate interval,
and let $x = G^{-1}(Y)$, where $G(x) = \int^{x} g(x')dx'$.

This method is applicable only when the function $G(x)$ is calculable and invertible. Happily, this is the case for the angular distribution $dN/d\theta$ of Eq. (2). Taking Y as a (negative) uniform random variate,

$$\theta = \sqrt{-2\sigma^2 \ln(-Y/\sigma^2)}. \qquad (3)$$

The limits on Y are given by $Y_1 = -\sigma^2$ and $Y_1 = -\sigma^2 \exp(-\theta_{max}{}^2/2\sigma^2)$ (corresponding to $\theta = 0$ and θ_{max} respectively).

2. Exponential distribution

Although the angular distribution of θ_e is well approximated by $d\sigma/d\Omega_e = Ae^{-b\theta}$, this would correspond to the probability distribution $dN/d\theta \propto \theta\, e^{-b\theta}$, whose integral is **not** readily invertible. Instead, if we take

$$d\sigma/d\Omega_e = A/\theta\; e^{-b\theta}, \qquad (4)$$

then $dN/d\theta \propto \theta\, d\sigma/d\Omega \propto e^{-b\theta}$ and we can take

$$\theta = -1/b \ln(-bY) \qquad (5)$$

where Y is a negative uniform random variate in the interval

$$-1/b \exp(-b\,\theta_{min}) < Y < -1/b \exp(-b\,\theta_{min}) \quad .$$

D. Monte Carlo implementation and results

Using the approximate fits to the calculated angular distribution of Section B, all that is required is to calculate the integral of each of the three functions over its region of applicability. Always taking θ in units of θ_{ce}, we can write (in arbitrary but consistent units):

	$d\sigma/d\Omega_e$	Normalized integral of $\theta\, d\sigma/d\Omega_e$ from θ_{min} to θ_{max}
$0 < \theta < 0.6$	$2.63 \exp(-\theta^2/2(0.55)^2)$	0.20211
$0.6 < \theta < 1.6$	$1.1748 \exp(-0.5\,\theta)/\theta$	0.39851 (cumulative: 0.60062)
$1.6 < \theta < 10$	$1.8986 \exp(-0.8\,\theta)/\theta$	0.37684 (cumulative: 0.97746)

Based on the considerations of Section C, the Monte Carlo procedure is:

1. Generate a uniform random variate $x \in [0,1]$.
2. If $x < 0.20211$, generate a URV $Y \in [-0.30250, -0.16684]$ and use Eq. (3) with $\sigma = 0.55$, which gives values of $\theta \in [0, 0.6]$.
3. Else if $x < 0.60062$, generate $Y \in [-1.48164, -0.89866]$ and use Eq. (5) with $b = 0.5$, giving values of $\theta \in [0.6, 1.6]$.
4. Else, generate $Y \in [-0.34755, -0.00041]$ and use Eq. (5) with $b = 0.8$, giving values of $\theta \in [1.6, 10]$.

A subroutine using this procedure was written and tested by generating 10^6 events and histogramming the angle θ (in units of θ_{ce}) in bins of width 0.1 θ_{ce} . Since the histogram generates angles according to the angular distribution dN/dθ, the number of events in each bin was divided by the central angle of each bin in order to give a distribution proportional to dN/dΩ (see Eq. (1) with $\sin\theta \approx \theta$.)

The histogram points (bin contents divided by θ), scaled by a single factor, are plotted in Figure 2 together with the calculated angular distribution $d\sigma/d\Omega_e$ for $E_e/E_0 = 0.2$ (solid curve) and the 3-part piecewise approximation (dashed curves). The agreement of the Monte Carlo distribution with the approximation is good – the small differences may be due to dividing by the central value of θ rather than an average value. The agreement with the calculated angular distribution is reasonable – probably good enough to use. I will not spend more time refining the calculation until I have had some comments on this Monte Carlo method and on the procedure outlined at the end of Section A.

References:

[Max88] Leonard C. Maximon, "Theoretical Aspects of Tagged Photon Facilities", presentation to conference, Moscow, December 1988 . There are two important corrections to the formulas in this paper: in Equations (13) and (14), the second term inside the { } bracket must begin with $4\varepsilon_1\varepsilon_2$, not $2\varepsilon_1\varepsilon_2$.

[Hub79] J.H. Hubbell and I. Øverbø, J.Phys.Chem.Ref.Data 8, (1979) 69.

[RPP14] Review of Particle Properties, 2014 Edition, K.A. Olive et al. (Particle Data Group), Chin. Phys. C, 38, 090001 (2014)

Table I. Comparison of contributions to variation of electron angle at the radiator, for E_0 = 12 GeV. The multiple scattering RMS angle was calculated assuming that the electron energy is E_0 for the first half of the target thickness and E_e for the second half; integrating in ten steps gives results 1-1.5% larger. For 20 µm and 50 µm diamond, $x/X_0 = 1.649\times10^{-4}$ and 4.122×10^{-4} respectively.

	Beam ellipse [mr]		Multiple scattering [mr]		θ_{ce} [mr]
E_e [GeV]	Horiz.	Vert.	20 µm	50 µm	
0.3	0.005	0.020	0.265	0.441	1.661
0.5	0.005	0.020	0.159	0.264	0.979
1	0.005	0.020	0.0796	0.133	0.468
1.5	0.005	0.020	0.0533	0.0889	0.298
2	0.005	0.020	0.0402	0.0671	0.213
2.5	0.005	0.020	0.0324	0.0540	0.162
3	0.005	0.020	0.0273	0.0454	0.128
3.5	0.005	0.020	0.0236	0.0394	0.103
4	0.005	0.020	0.0209	0.0349	0.085
4.5	0.005	0.020	0.0188	0.0314	0.071

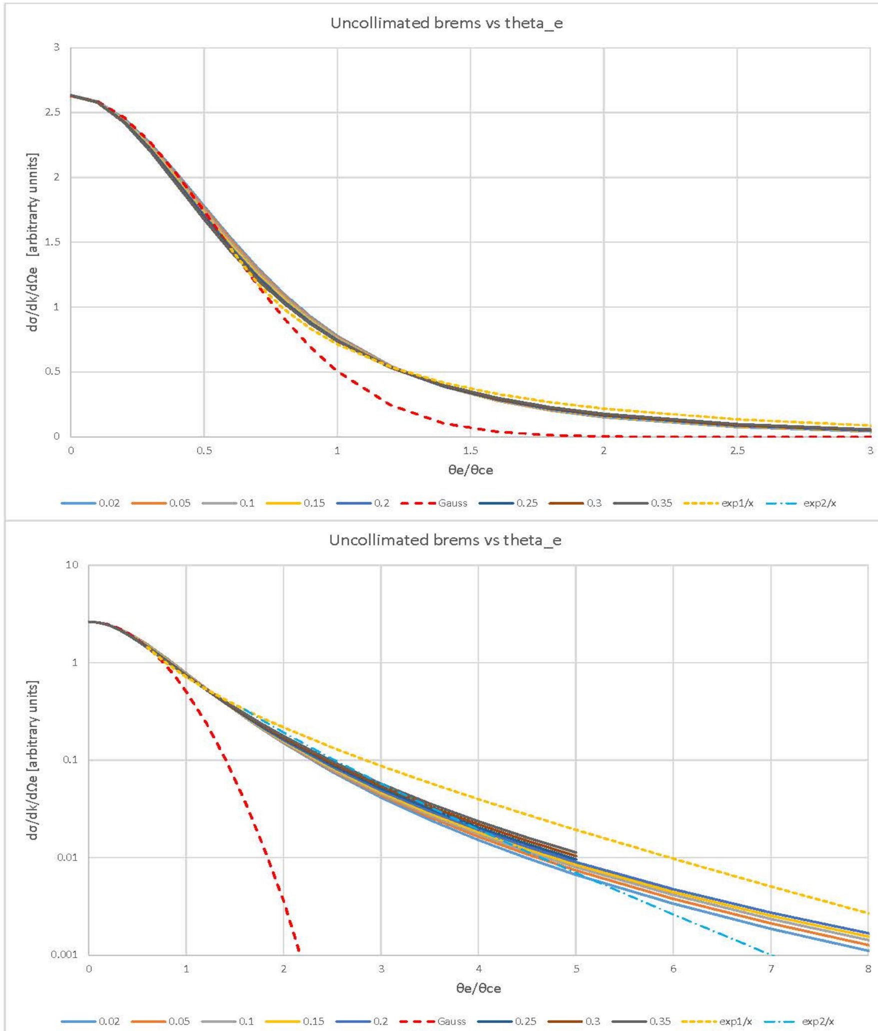

Figure 1. Solid curves: scaled angular distribution $d\sigma/dk\ d\Omega_e$ versus θ_e/θ_{ce} .
Dashed curves: Approximate fits to the angular distribution valid for the regions $\theta_e/\theta_{ce} < 0.6$, 0.6-1.6, and 1.6-10.

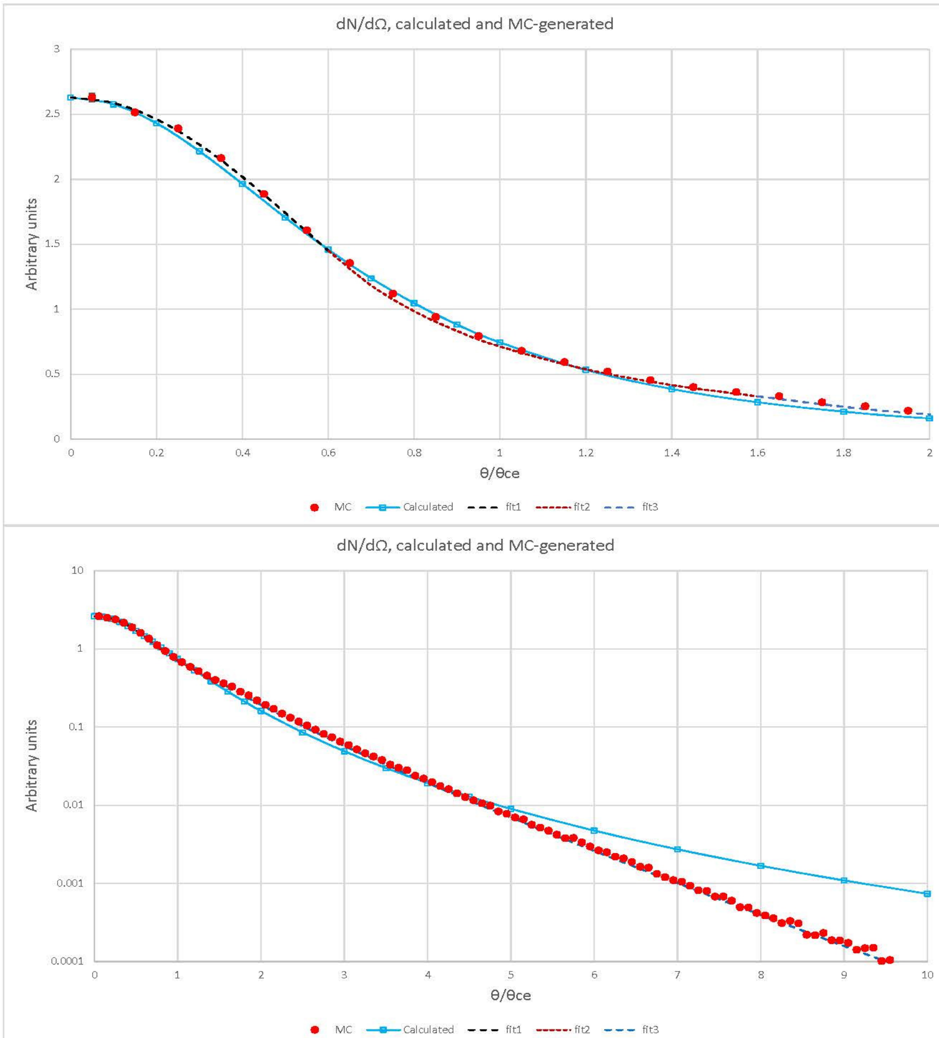

Figure 2. Solid curve: Calculated bremsstrahlung angular distribution dσ/dk dΩ_e for E_e/E_0 = 0.2 (other values very similar.) Dashed curves: Analytical functions of Section B in the 3 regions, 0-0.6, 0.6-1.6, 1.6-10. Points: Histogram of Monte-Carlo-generated distribution of dN/dθ divided by central value of θ for each bin.