GlueX Internal Communication # 4599

Comparing Formalisms for the Analysis of Two-pseudoscalar Mesons Produced by Linearly Polarized Photons

Carlos Salgado[1] and Vincent Mathieu[2]
Norfolk State U and JLab, USA[1]
and
Universidad Complutense de Madrid, Spain[2]

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The Photon Spin Density Matrix

Carlos Salgado
NSU and JLab

Abstract

We compare different formalisms of reflectivity, based in references ([9],[3] and [2]), used to study the production of two pseudo-scalar mesons with linearly polarized photons off the proton. One of the formalisms ("new reflectivity") is introduced for the first time in this note. We present the definitions and then compare the formalisms analytically and using Monte Carlo simulations.

Contents

The cross sections (intensity) should be independent of the basis chosen to represent their amplitudes. Experimentally, however, we use those intensities to fit data and obtain parameters for the phenomenological models. In practice, different analytical choices for representing the same intensity can give different results from a fitting procedure (see as an example the study in reference [1]). It is, then, of critical importance to compare different mathematical representations.

$$I(\Omega,\mathscr{P},\Phi)=\sum_{\lambda_1\lambda_2}\sum_{i,j}\sum_{l,m,l',m'} A_{l,m}(\Omega)\,{}^{i}T^{\lambda_1,\lambda_2}_{l,m}\,\rho^{\gamma}_{ij}(\mathscr{P},\Phi)\left[{}^{j}T^{\lambda_1,\lambda_2}_{l',m'}\right]^{*} A^{*}_{l',m'}(\Omega). \quad (15)$$

The Helicity Basis

$$\boxed{I(\Omega,\mathscr{P},\Phi)=\sum_{\lambda_1\lambda_2}\sum_{\lambda,\lambda'}\sum_{l,m,l',m'} Y^{m}_{l}(\Omega)\,{}^{\lambda}T^{\lambda_1\lambda_2}_{l,m}\,\rho^{\gamma}_{\lambda,\lambda'}(\mathscr{P},\Phi)\left[{}^{\lambda'}T^{\lambda_1\lambda_2}_{l',m'}\right]^{*} Y^{m'*}_{l'}(\Omega)}. \quad (20)$$

Using equations (64) and (67), and in the helicity basis, $P^{j}_{\gamma}=P_{\gamma}(-cos2\Phi,-sin2\Phi,0)$ [4]:

$$I(\Omega,\mathscr{P},\Phi)=I^{0}-I^{1}P_{\gamma}cos2\Phi-I^{2}P_{\gamma}sin2\Phi \quad (31)$$

with

$$I^{0}(\Omega)=\sum_{\lambda,\lambda_1\lambda_2}\sum_{l,m,l',m'} {}^{\lambda}T^{\lambda_1\lambda_2}_{l,m}\left[{}^{\lambda}T^{\lambda_1\lambda_2}_{l',m'}\right]^{*}Y^{m}_{l}(\Omega)\;Y^{m'*}_{l'}(\Omega) \quad (32)$$

.

$$I^{1}(\Omega)=\sum_{\lambda,\lambda_1\lambda_2}\sum_{l,m,l',m'} {}^{-\lambda}T^{\lambda_1\lambda_2}_{l,m}\left[{}^{\lambda}T^{\lambda_1\lambda_2}_{l',m'}\right]^{*}Y^{m}_{l}(\Omega)\;Y^{m'*}_{l'}(\Omega) \quad (33)$$

.

$$I^{2}(\Omega)=i\cdot\sum_{\lambda,\lambda_1\lambda_2}\sum_{l,m,l',m'} \lambda\cdot{}^{-\lambda}T^{\lambda_1\lambda_2}_{l,m}\left[{}^{\lambda}T^{\lambda_1\lambda_2}_{l',m'}\right]^{*}Y^{m}_{l}(\Omega)\;Y^{m'*}_{l'}(\Omega) \quad (34)$$

The (Old) Reflectivity Operator

$$\widehat{\Pi}|Jm\rangle = P(-1)^{J-m}e^{i\pi J_y}|J-m\rangle.$$

It is useful to define the reflection operator [9]

$$\widehat{\Pi}_y = \widehat{\Pi}e^{-i\pi J_y}$$

$$|\epsilon, l, m\rangle = \left[|l, m\rangle - \epsilon P(-1)^{(l-m)}|l, -m\rangle\right]\Theta(m)$$

where

$$\Theta(m) = \frac{1}{\sqrt{2}},\ if\ m > 0$$
$$\Theta(m) = \frac{1}{2},\ if\ m = 0$$
$$\Theta(m) = 0,\ if\ m < 0$$

$$|\epsilon_R, l, |m|\rangle = \left[|l, m\rangle - \epsilon_R(-1)^m|l, -m\rangle\right]\Theta(m)$$

$$^{\epsilon_R}Y_l^{|m|} = \left[Y_l^m - \epsilon_R(-1)^m Y_l^{-m}\right]\Theta(m)$$

$$\boxed{I(\Omega) = \sum_k \sum_{\epsilon_R} \sum_{l,|m|,l',|m'|} {}^{\epsilon_R}Y_l^{|m|}(\Omega)\ {}^{\epsilon_R}T^k_{l,|m|}\ {}^{\epsilon_R}T^{k*}_{l',|m'|}\ {}^{\epsilon_R}Y_{l'}^{|m'|*}(\Omega)} \qquad (55)$$

The New Reflectivity Basis

$$|\epsilon_\gamma, \lambda\rangle = \left[|\lambda\rangle - \epsilon_\gamma(-1)^\lambda| - \lambda\rangle\right]\Theta(\lambda)$$

then (the reflectivity eigenvalues for a photon are $\epsilon_\gamma = \pm 1$).

$$^{\epsilon_R}Y_l^{|m|} = \left[Y_l^m - \epsilon_R(-1)^m Y_l^{-m}\right]\Theta(m)$$

$$|\epsilon_\gamma = +1, \lambda = +1\rangle = \frac{1}{\sqrt{2}}(|\lambda = +1\rangle + |\lambda = -1\rangle)$$

$$|\epsilon_\gamma = -1, \lambda = +1\rangle = \frac{1}{\sqrt{2}}(|\lambda = +1\rangle - |\lambda = -1\rangle)$$

.

$$\rho_{\epsilon_\gamma, \epsilon'_\gamma}(\mathscr{P}, \Phi) = 1/2 \begin{pmatrix} 1 - \mathscr{P}\cos 2\Phi & -i\mathscr{P}\sin 2\Phi \\ i\mathscr{P}\sin 2\Phi & 1 + \mathscr{P}\cos 2\Phi \end{pmatrix}$$

$$\boxed{I(\Omega, \mathscr{P}, \Phi) = \sum_k \sum_{\epsilon_\gamma, \epsilon'_\gamma} \sum_{\epsilon_R, \epsilon'_R} \sum_{l|m|, l'|m'|} {}^{\epsilon_R}Y_l^{|m|}(\Omega)\, {}^{\epsilon_R \epsilon_\gamma}T^k_{l,|m|}\, \rho^\gamma_{\epsilon_\gamma, \epsilon'_\gamma}(\mathscr{P}, \Phi)\, {}^{\epsilon'_R \epsilon'_\gamma}T^{k*}_{l',|m'|}\, {}^{\epsilon'_R}Y_{l'}^{|m'|*}(\Omega)}. \tag{61}$$

Using equations (64) and (67), and in the reflectivity basis, $P^j_\gamma = P_\gamma(0, sin2\Phi, -cos2\Phi)$ (see [4]):

$$I(\Omega, \mathscr{P}, \Phi) = I^0 + I^2 P_\gamma sin2\Phi - I^3 P_\gamma cos2\Phi \tag{72}$$

with

$$I^0(\Omega) = \sum_k \sum_{\epsilon_\gamma, \epsilon_R \epsilon'_R} \sum_{l,|m|,l',|m'|} {}^{\epsilon_\gamma \epsilon_R}T^k_{l,|m|}\left[{}^{\epsilon_\gamma \epsilon'_R}T^k_{l',|m'|}\right]^* {}^{\epsilon_R}Y_l^{|m|}(\Omega)\, {}^{\epsilon'_R}Y_{l'}^{|m'|*}(\Omega) \tag{73}$$

.

$$I^2(\Omega) = i \cdot \sum_k \sum_{\epsilon_\gamma, \epsilon_R \epsilon'_R} \sum_{l,|m|,l',|m'|} \epsilon_\gamma \cdot {}^{-\epsilon_\gamma \epsilon_R}T^k_{l,|m|}\left[{}^{\epsilon_\gamma \epsilon'_R}T^k_{l',|m'|}\right]^* {}^{\epsilon_R}Y_l^{|m|}(\Omega)\; {}^{\epsilon'_R}Y_{l'}^{|m'|*}(\Omega) \tag{74}$$

.

$$I^3(\Omega) = \sum_k \sum_{\epsilon_\gamma, \epsilon_R \epsilon'_R} \sum_{l,|m|,l',|m'|} \epsilon_\gamma \cdot {}^{\epsilon_\gamma \epsilon_R}T^k_{l,|m|}\left[{}^{\epsilon_\gamma \epsilon'_R}T^k_{l',|m'|}\right]^* {}^{\epsilon_R}Y_l^{|m|}(\Omega)\; {}^{\epsilon'_R}Y_{l'}^{|m'|*}(\Omega) \tag{75}$$

.

The JPAC Reflectivity Basis

$$I(\Omega, \mathscr{P}, \Phi) = \sum_{\lambda_1,\lambda_2} \sum_{\lambda,\lambda'} \sum_{lm,l'm'} T^{l}_{\lambda,m;\lambda_1,\lambda_2} Y^{m}_{l}(\Omega) \rho^{\gamma}_{\lambda,\lambda'}(\mathscr{P}, \Phi) T^{l'*}_{\lambda',m';\lambda_1,\lambda_2} Y^{m'*}_{l'}(\Omega).$$

$$\rho^{\gamma}_{\lambda\lambda'}(\mathscr{P}, \Phi) = 1/2 \begin{pmatrix} 1 & -\mathscr{P}e^{-2i\Phi} \\ -\mathscr{P}e^{2i\Phi} & 1 \end{pmatrix}$$

$${}^{(\epsilon)}T^{l}_{m;\lambda_1,\lambda_2} = \frac{1}{2}\left[T^{l}_{\lambda=+1,m;\lambda_1,\lambda_2} - \epsilon(-1)^{m} T^{l}_{\lambda=-1,-m;\lambda_1,\lambda_2}\right]$$

$$\boxed{I(\Omega, \mathscr{P}, \Phi) = I^{(0)}(\Omega) - \mathscr{P} I^{(1)}(\Omega)\cos 2\Phi - \mathscr{P} I^{(2)}(\Omega)\sin 2\Phi}$$

$$I^{(0)}(\Omega) = \sum_{\epsilon,k} \left|U^{(\epsilon)}{}_{k}(\Omega)\right|^{2} + \left|\tilde{U}^{(\epsilon)}{}_{k}(\Omega)\right|^{2}$$

$$I^{(1)}(\Omega) = -\sum_{\epsilon,k} 2\epsilon Re\left(U^{(\epsilon)}{}_{k}(\Omega)\left[\tilde{U}^{(\epsilon)}{}_{k}(\Omega)\right]^{*}\right)$$

$$I^{(2)}(\Omega) = -\sum_{\epsilon,k} 2\epsilon Im\left(U^{(\epsilon)}{}_{k}(\Omega)\left[\tilde{U}^{(\epsilon)}{}_{k}(\Omega)\right]^{*}\right)$$

with

$$U^{(\epsilon)}{}_{k}(\Omega) = \sum_{l,m} [l]^{\epsilon}{}_{m,k} Y^{m}_{l}(\Omega)$$

$$\tilde{U}^{(\epsilon)}{}_{k}(\Omega) = \sum_{l,m} [l]^{\epsilon}{}_{m,k} \left[Y^{m}_{l}(\Omega)\right]^{*}$$

Naturality and Reflectivity

Definition: $$\mathcal{N} = P \times (-1)^J.$$

In the JPAC definition, by equation (35):

$$\epsilon = P(-1)^J$$

or

$$\boxed{\epsilon = \mathcal{N}}$$

$$\epsilon_{beam} \times \epsilon_{ex} = \epsilon_R.$$

or

$$\epsilon_{photon} \times \epsilon_{resonance} = \epsilon_{exchange}.$$

And by definition $\epsilon_{exchange} = P(-1)^J$, then

$$\boxed{\epsilon_{photon} \times \epsilon_{resonance} = \mathcal{N}.}$$

Comparing Formalisms

In summary, all three bases are equivalent for calculating the intensity in the partial wave expansion, and the transformation relations are:
(A) from Helicity → New reflectivity.

$$ {}^{\epsilon_R\epsilon_\gamma}T^{\ell}_{m;\lambda_1\lambda_2} = \frac{\theta(m)}{\sqrt{2}}\left(T^{\ell}_{1m;\lambda_1\lambda_2} + \epsilon_\gamma T^{\ell}_{-1m;\lambda_1\lambda_2} - \epsilon_R(-1)^m T^{\ell}_{1-m;\lambda_1\lambda_2} - \epsilon_\gamma\epsilon_R(-1)^m T^{\ell}_{-1-m;\lambda_1\lambda_2}\right) \tag{143} $$

(B) from Helicity → JPAC.

$$ {}^{(\epsilon)}T^{l}_{m;\lambda_1,\lambda_2} = \frac{1}{2}\left[T^{l}_{\lambda=+1,m;\lambda_1,\lambda_2} - \epsilon(-1)^m T^{l}_{\lambda=-1,-m;\lambda_1,\lambda_2}\right] \tag{144} $$

(C) from JPAC reflectivity → New reflectivity.

$$ {}^{\pm\pm}T^{\ell}_{m;\lambda_1\lambda_2} = \sqrt{2}\theta(m)\left[{}^{(+)}T^{\ell}_{m;\lambda_1\lambda_2} \mp (-1)^m\, {}^{(+)}T^{\ell}_{-m;\lambda_1\lambda_2}\right] \tag{145a} $$

$$ {}^{\pm\mp}T^{\ell}_{m;\lambda_1\lambda_2} = \sqrt{2}\theta(m)\left[{}^{(-)}T^{\ell}_{m;\lambda_1\lambda_2} \mp (-1)^m\, {}^{(-)}T^{\ell}_{-m;\lambda_1\lambda_2}\right] \tag{145b} $$

7.1 Unpolarized Intensity

Only P-wave

Both give:

$$ I(\Omega) = \frac{3}{4\pi}\left[\rho_{00}\cos^2\theta + \rho_{11}\sin^2\theta - \sqrt{2}\,\mathrm{Re}\,\rho_{10}\sin 2\theta\cos\phi - \rho_{1-1}\sin^2\theta\cos 2\phi\right] $$

Polarized Intensity

$$ I(\Omega,P_\gamma,\Phi) = \kappa\sum_{\epsilon,k}\sum_{\substack{\ell,m\\ \ell',m'}}[\ell]^{(\epsilon)}_{m;k}Y^m_\ell(\Omega)\left[1+(-1)^{m-m'} - P_\gamma(-1)^{m'}e^{-2i\Phi} - P_\gamma(-1)^m e^{2i\Phi}\right][\ell']^{(\epsilon)*}_{m';k}Y^{m'*}_{\ell'}(\Omega) $$

$$ \begin{aligned} I(\Omega,P_\gamma,\Phi) = &\sum_{\substack{\ell,m\\ \ell',m'}}\sum_{\lambda,\lambda'} T^{\ell}_{\lambda m;++}Y^m_\ell(\Omega)\rho_{\lambda,\lambda'}(P_\gamma,\Phi)T^{\ell*}_{\lambda' m';++}Y^{m'*}_{\ell'}(\Omega) \\ &+\sum_{\substack{\ell,m\\ \ell',m'}}\sum_{\lambda,\lambda'} T^{\ell}_{\lambda m;--}Y^m_\ell(\Omega)\rho_{\lambda,\lambda'}(P_\gamma,\Phi)T^{\ell*}_{\lambda' m';--}Y^{m'*}_{\ell'}(\Omega) \\ &+\sum_{\substack{\ell,m\\ \ell',m'}}\sum_{\lambda,\lambda'} T^{\ell}_{\lambda m;+-}Y^m_\ell(\Omega)\rho_{\lambda,\lambda'}(P_\gamma,\Phi)T^{\ell*}_{\lambda' m';+-}Y^{m'*}_{\ell'}(\Omega) \\ &+\sum_{\substack{\ell,m\\ \ell',m'}}\sum_{\lambda,\lambda'} T^{\ell}_{\lambda,m;-+}Y^m_\ell(\Omega)\rho_{\lambda,\lambda'}(P_\gamma,\Phi)T^{\ell*}_{\lambda' m';-+}Y^{m'*}_{\ell'}(\Omega) \end{aligned} $$

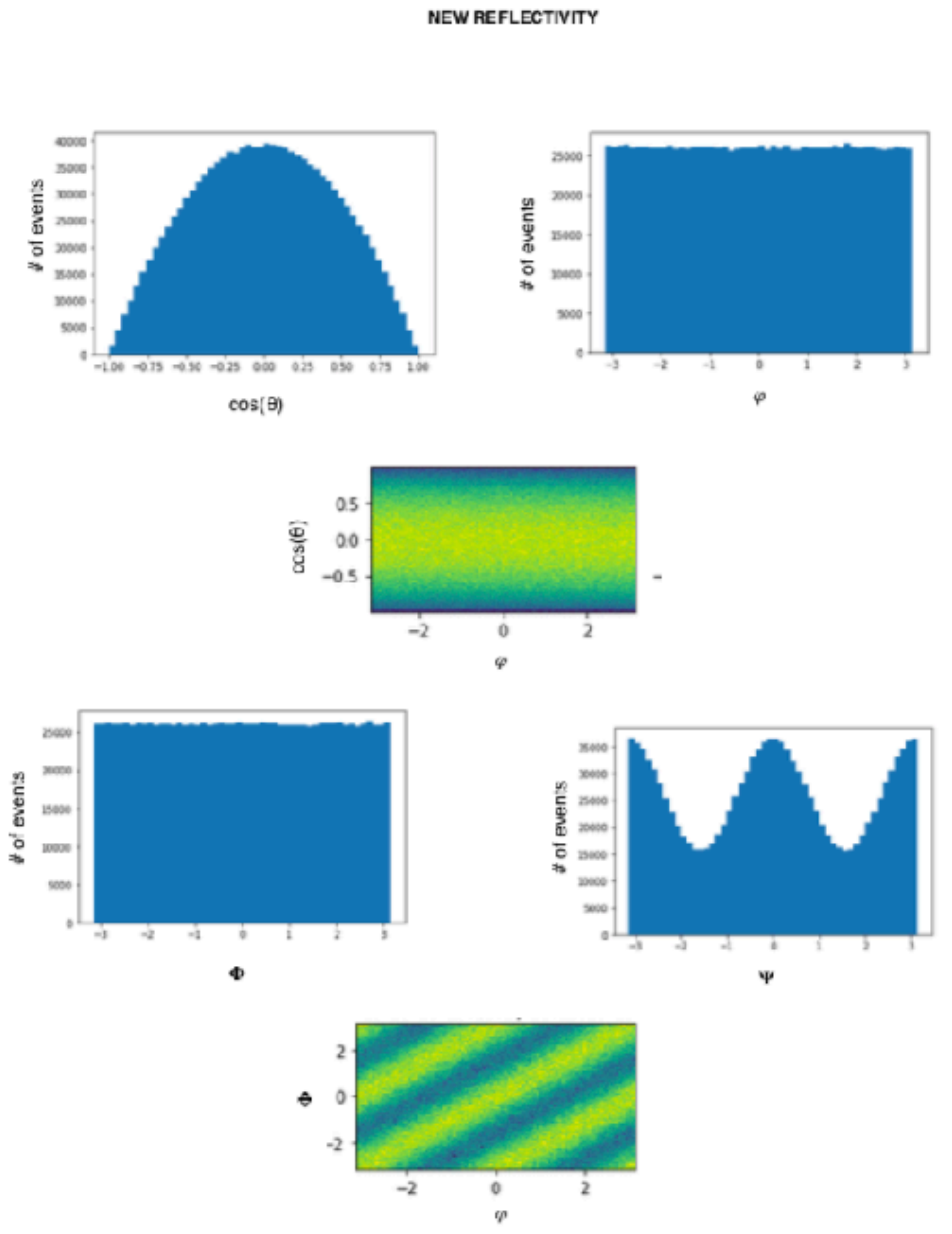
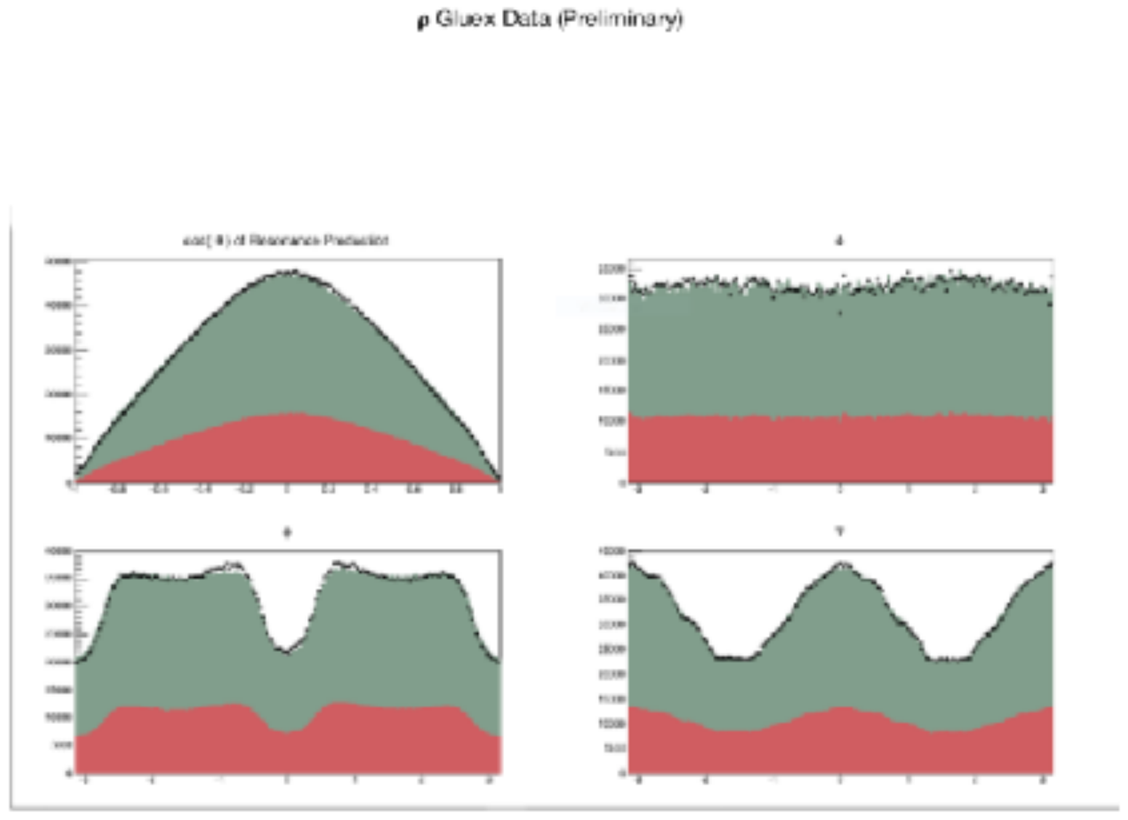
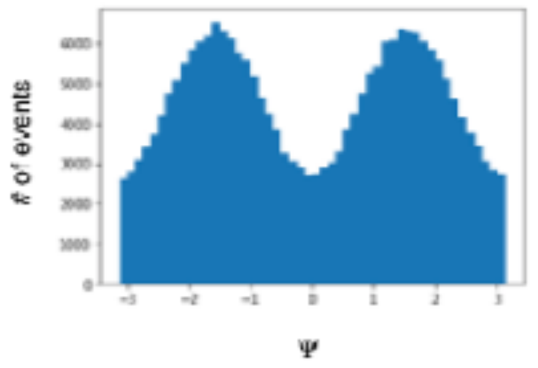


Figure 1: NEW-REF distributions from simulation generated for π_1 . Only natural exchange (naturality =+1) and P waves with ($\epsilon = -1, \epsilon R = -1, L = 1, M = 1$) and ($\epsilon = 1, \epsilon R = 1, L = 1, M = 1$) (both with same weights).



Relation between bases

$$|-,-,1,1\rangle + |+,+,1,1\rangle \equiv |+,1,1\rangle$$



Unnatural

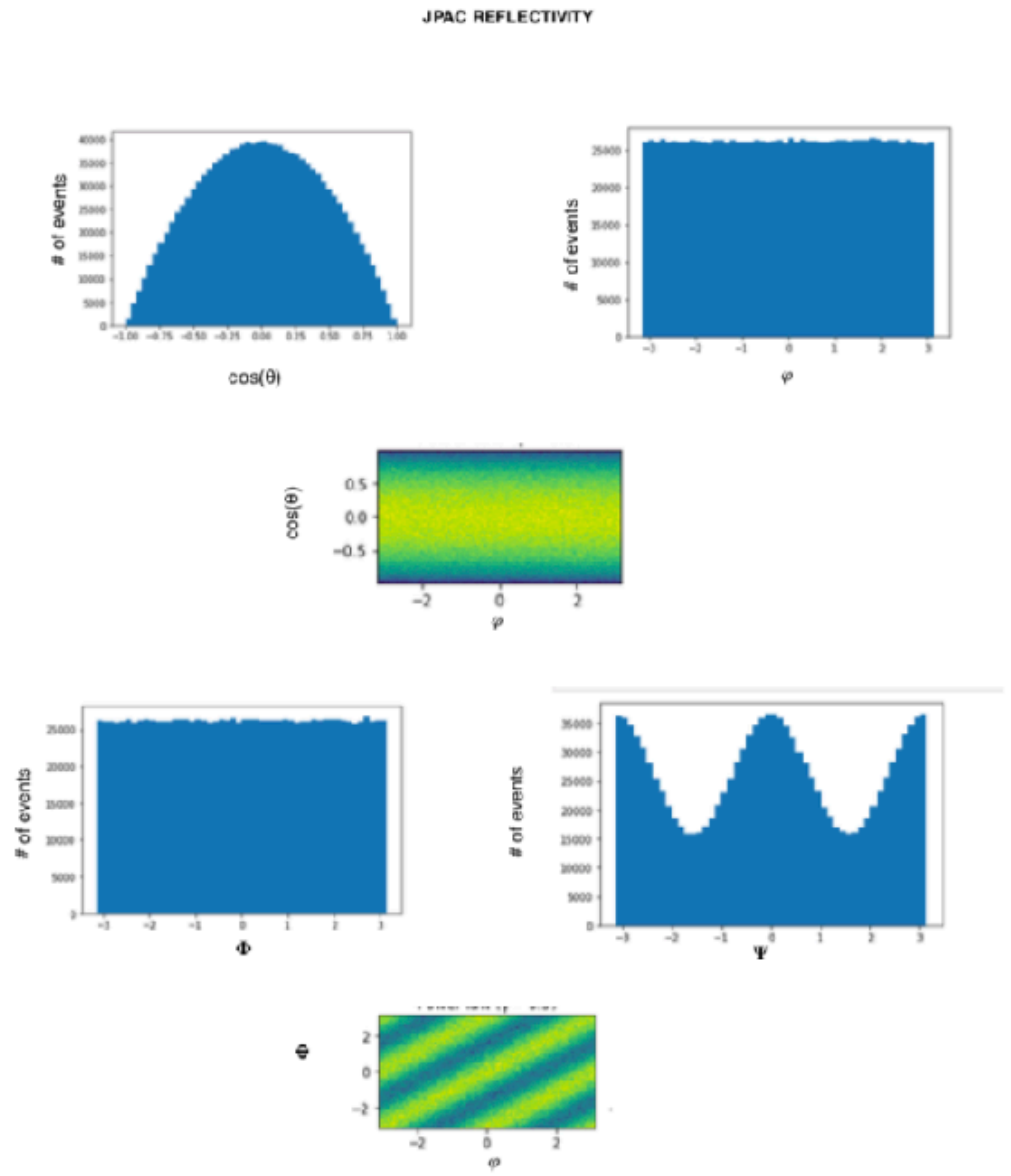


Figure 2: JPAC-REF distributions from simulation generated for π_1 . Only natural exchange (naturally =+1) and P waves with ($\epsilon = 1, L = 1, M = 1$)

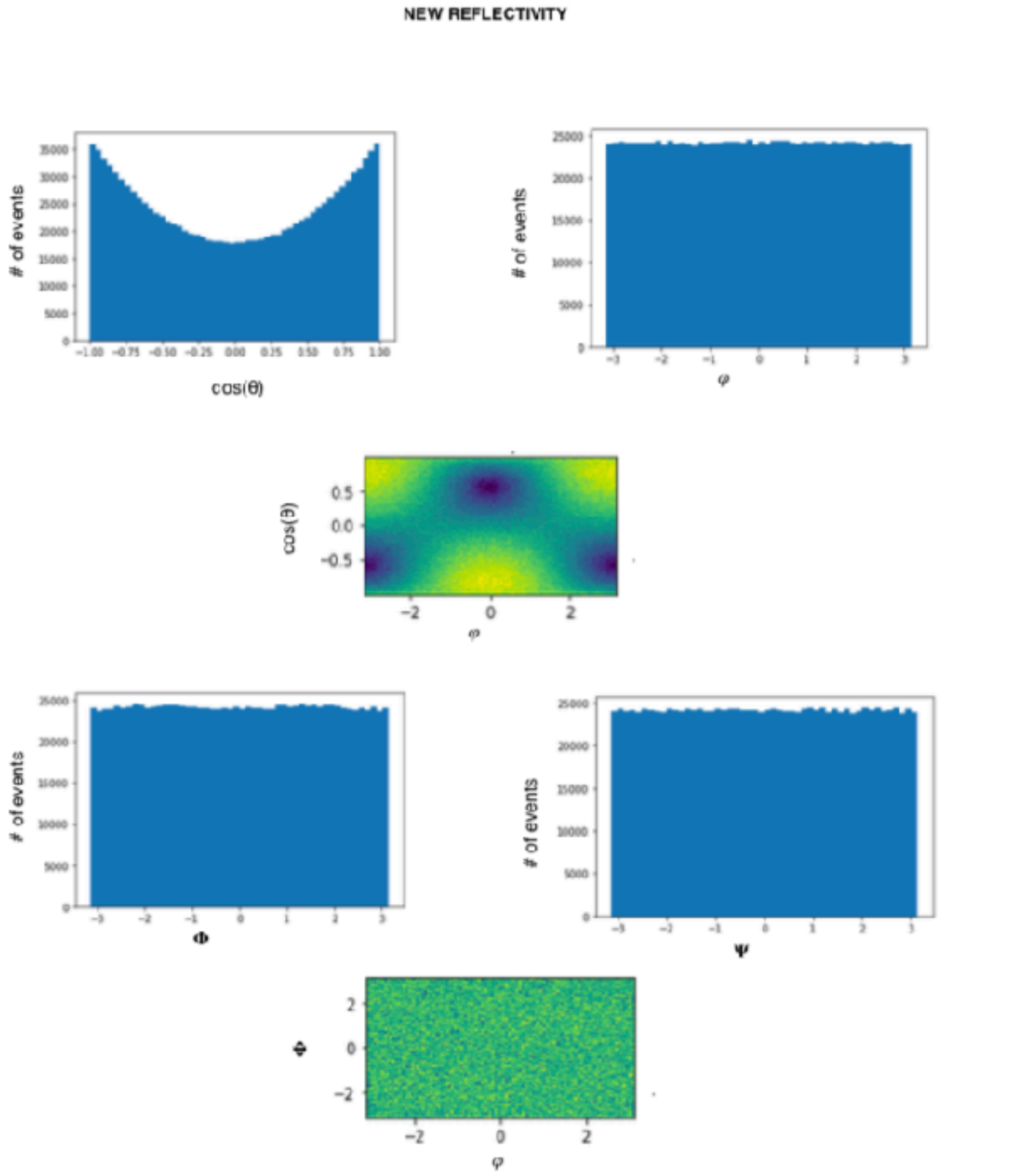


Figure 5: NEW-REF distributions from simulation generated for π_1. All possible P waves (see text) with same weights.

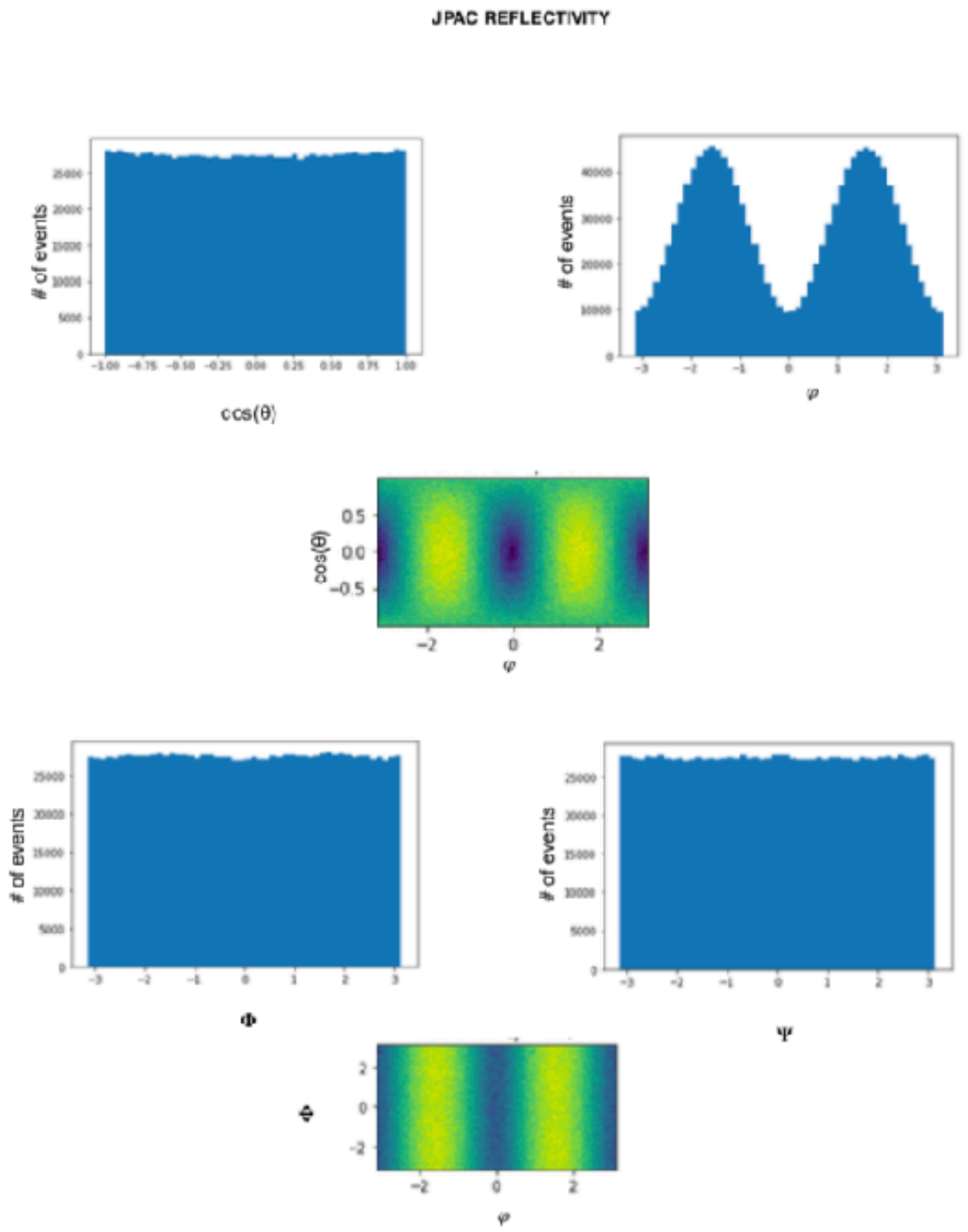


Figure 6: JPAC-REF distributions from simulation generated for π_1. All possible P waves (see text) with same weights.

$$
\begin{aligned}
|-,-,1,0\rangle &= \frac{1}{\sqrt{2}}(|+,1,0\rangle + |+,1,0\rangle) = \sqrt{2}|+,1,0\rangle \\
|-,-,1,1\rangle &= |+,1,1\rangle - |+,1,-1\rangle \\
|-,+,1,1\rangle &= |-,1,1\rangle - |-,1,-1\rangle \\
|+,-,1,0\rangle &= \frac{1}{\sqrt{2}}|-,1,0\rangle - |-,1,0\rangle \\
|+,-,1,1\rangle &= |-,1,1\rangle - |-,1,-1\rangle \\
|+,+,1,1\rangle &= |+,1,1\rangle + |+,1,-1\rangle
\end{aligned}
$$

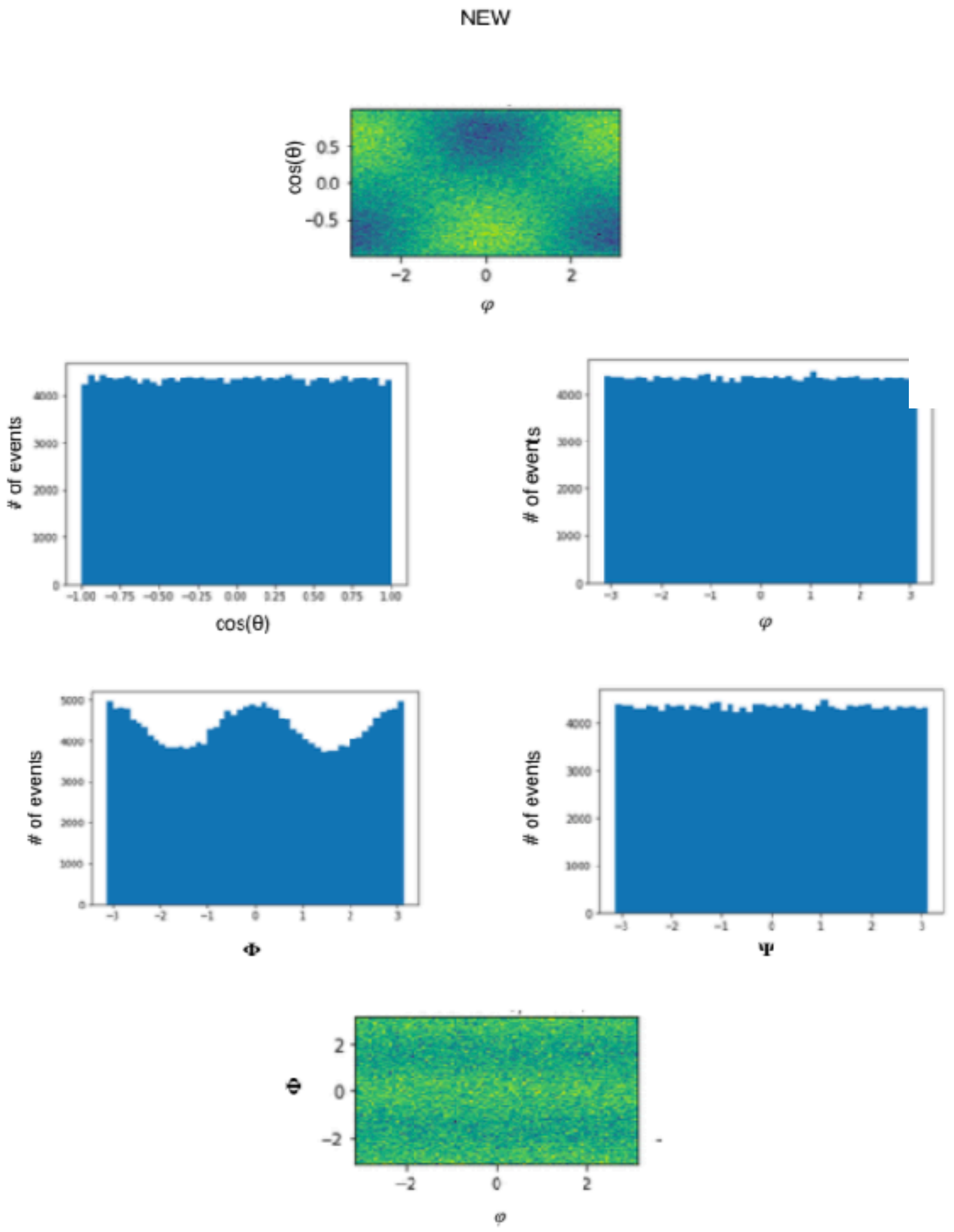


Figure 7: NEW-REF distributions from simulation generated for π_1. All possible P waves (see text) with weights balanced according to the transformations between bases.

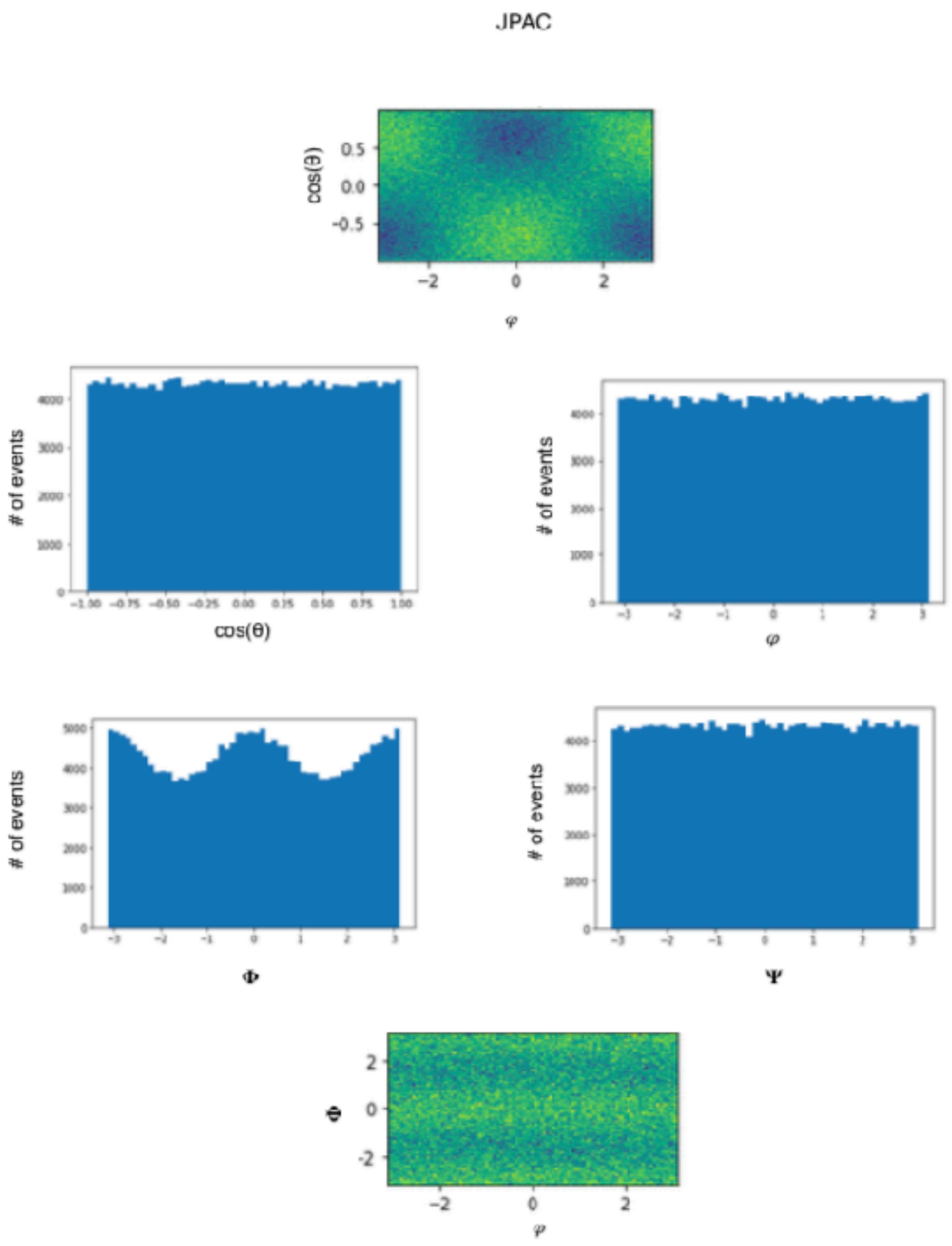


Figure 8: JPAC-REF distributions from simulation generated for π_1. All possible P waves (see text) with weights balanced according to the transformations between bases.

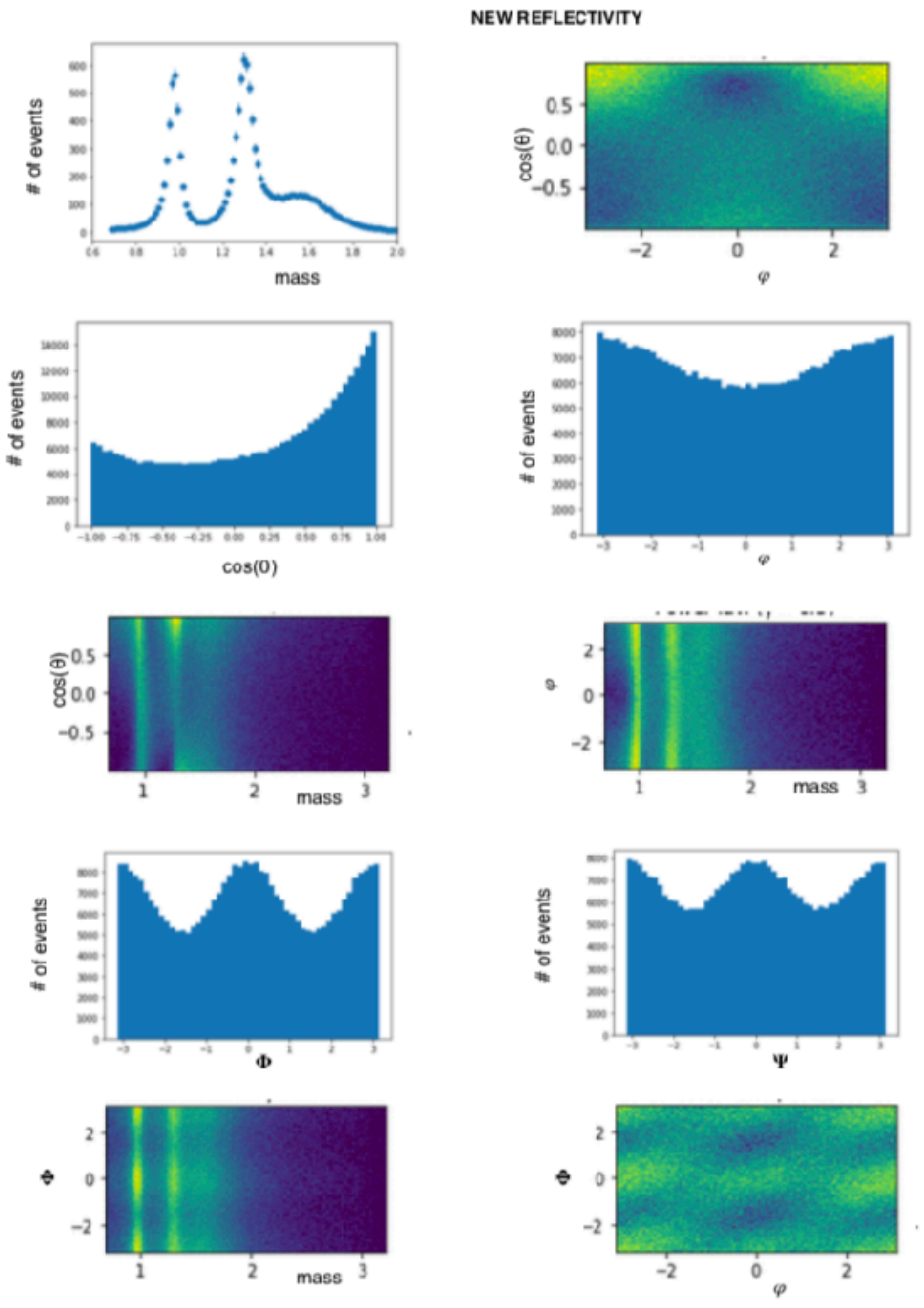


Figure 9: NEW-REF distributions from simulation generated for three resonances at a_0 in S wave, a_2 in D waves and π_1 in P waves. Only natural exchange and $|m| < 2$. See text for wave's description.

(naturally = +1) produced a_0 , $a_2$and π_1 using the following waves: for the NEW:

$$\epsilon_\gamma = -1, \epsilon_R = -1, L = 0, M = 0; a_0(980) \tag{192}$$
$$\epsilon_\gamma = -1, \epsilon_R = -1, L = 2, M = 0; a_2(1306) \tag{193}$$
$$\epsilon_\gamma = 1, \epsilon_R = +1, L = 2, M = 1; a_2(1306) \tag{194}$$
$$\epsilon_\gamma = -1, \epsilon_R = -1, L = 2, M = 1; a_2(1306) \tag{195}$$
$$\epsilon_\gamma = -1, \epsilon_R = -1, L = 1, M = 0; \pi_1(1564) \tag{196}$$
$$\epsilon_\gamma = 1, \epsilon_R = +1, L = 1, M = 1; \pi_1(1564) \tag{197}$$
$$\epsilon_\gamma = -1, \epsilon_R = -1, L = 1, M = 1; \pi_1(1564) \tag{198}$$
$$\tag{199}$$

Using the transformation relations, we can find that those correspond in the JPAC:

$$\epsilon = 1, L = 0, M = 0; a_0(980) \tag{200}$$
$$\epsilon = 1, L = 2, M = 1; a_2(1306) \tag{201}$$
$$\epsilon = 1, L = 2, M = 0; a_2(1306) \tag{202}$$
$$\epsilon = 1, L = 1, M = 1; \pi_1(1564) \tag{203}$$
$$\epsilon = 1, L = 1, M = 0; \pi_1(1564) \tag{204}$$
$$\tag{205}$$

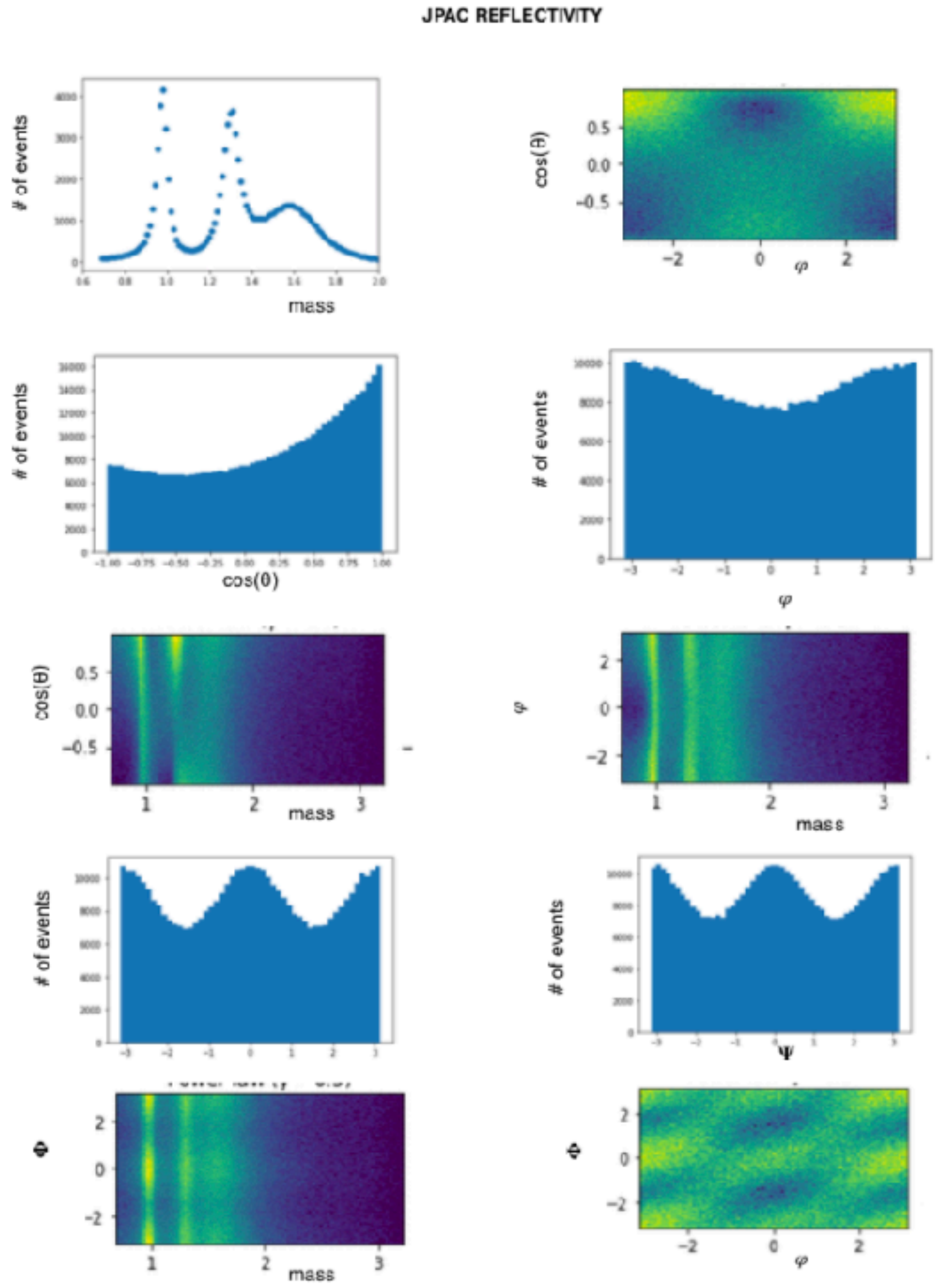


Figure 10: JPAC-REF distributions from simulation generated for three resonances at a_0 in S wave, a_2 in D waves and π_1 in P waves. Only natural exchange and $|m| < 2$. See text for wave's description.

Summary

- Both formalisms have the same number of degrees of freedom (parameters) per wave. These are 2 x 2 x (2L+1).
- Both definitions produce equivalent distributions for the azimuthal and polar angles (ϕ, θ) and also polarization related angles Φ and Ψ. The values of the intensity $(I(\Omega))$ in both definitions are equivalent (if the transformation between the bases is applied).
- A mass independent PWA is basically a fit to the final particle's angular distributions (ϕ, θ) (in a mass independent fit). Since angular distributions are represented by different internal weights in both formalisms, the different formalism can produce different sensitivity to angular distributions on the fitted parameters. The use of different analytical functions, initial fit values, optimization techniques, and even binning of data and analysis cuts have been shown to bias fitting results in a complex non-linear analysis such PWA (i.e. see [16]. (In two pseudo-scalar analysis we can add physical ambiguities and false optimizations (minima)). Therefore both formalisms will need to be evaluated in PWA fits. We plan to study these possible effects in the future using MC-simulated and real data.